Sealed-Bid Auctions

An object has to be assigned to one player, i.e., \( u_1, ..., u_n \) in exchange for a payment. For each player \( i \), \( u_i \) is the valuation of player \( i \) of the object. That is, we assume that \( u_1 > u_2 > u_3 > ... > u_n \).

Mechanism: Players simultaneously give their bids \( b_1, b_2, ..., b_n \geq 0 \). The object is given to the bidder \( i \) with maximal bid \( b_i \). Break ties by valuation order, i.e., if \( b_i = b_j \) and the highest bids, then \( i \) will win off \( i < j \).
First price auction: The payment by the winner is the highest bid.

Second price auction: The payment by the winner is the highest bid of non-empty bidders.

Formulation:

\[ N = \{1, \ldots, n\} \]

\[ A_i = \{ b_i \mid b_i \in \mathbb{R}_+ \} \]

\[ u_i(b) = \begin{cases} 0 & \text{if player } i \text{ does not win} \\ u_i - b_i & \text{otherwise} \end{cases} \]
For second-price auction:

\[ u_i(b) = \begin{cases} 
0, & \text{if } i \text{ plays } 1 \text{ and } u_i > u_j \\
u_i - \max b_{-i}, & \text{otherwise.}
\end{cases} \]

Example: Three bidders 1, 2, 3.

\[ u_1 = 100, \quad u_2 = 80, \quad u_3 = 53 \]
\[ b_1 = 90, \quad b_2 = 85, \quad b_3 = 45 \]

Bidder 1 wins both types of auctions.

First-price auction: \[ u_1(b) = u_1 - b_1 = 100 - 90 = 10. \]
Second-price auction: \[ u_1(b) = u_1 - b_2 = 100 - 85 = 15. \]
Proposition: In a second-price auction, bidding your own valuation, \( b_i \), is a weakly dominant strategy.

Proof: 1) Regardless of what the other bidders do, \( b_i \) is always a best response strategy.

Case I) \( i \) wins: \( i \) has to pay \( \max b_{-i} \leq v_i \), which means that \( u_i (b_{-i}, b_i) \geq 0 \).

Case I.1) \( i \) decreases his bid: does not help (\( i \) will still win at the same payment, or will lose and pay \( v_i \)).

Case I.2) \( i \) increases his bid: \( i \) still wins, pays the same amount as before.
Case II) i (cass). $u_i (b_{-i}, b_i^+) = 0.$

Case II.1: i decreases his bid:
- If i loses, $u_i$ utility 0.

Case II.2: i increases his bid:
- If i still loses, $u_i$ utility 0;
- If i becomes winner, i pays more than the object is worth to him =) negative utility.
2) \( b^+_i \) is strictly better than any other strategy under some opponent profile \( b_{-i} \).

Let \( b'_i \) be some strategy \( \neq b^+_i \).

Case I) \( b'_i < b^+_i \). Now let us consider \( b_{-i} \) with \( b^+_i > \max b_{-i} > b'_i \). With \( b'_i \), we do not win any more, i.e., we have
\[
u_i (b_{-i}, b'_i) = 0,
\]
whereas
\[
u_i (b_{-i}, b^+_i) > 0.
\]

Case II) \( b'_i > b^+_i \). Consider \( b'_i > \max b_{-i} > b^+_i \).

Here \( \nu_i (b_{-i}, b'_i) < 0 \), but \( \nu_i (b_{-i}, b^+_i) > 0 \). \(\Box\)
Remark: A profile of weakly dominant strategies is a NE, because for no player there is an incentive to deviate to a different action.

Remark: There are other NE, as well.
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<tbody>
<tr>
<td>L</td>
<td>2, 1</td>
<td>2, -20</td>
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<tr>
<td>R</td>
<td>-100, 2</td>
<td>3, 3</td>
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Pl. 1

\[ \rightarrow \text{2} \]

\[ \rightarrow -10 \]

\[ \rightarrow -1000 \]
Two-Sum Games and NE

Def.: A two-sum game (TSG) is a 2-player strategic game

\[ G = \langle \{1,2\}, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \]

\[ N \]

such that for all profiles \( a \in A \): \( u_1(a) + u_2(a) = 0 \).

Remark: Can be generalized to constant-sum games, where the utilities sum up to a constant C.
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<td>N</td>
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<td>B</td>
<td>-6, 6</td>
<td>4, -4</td>
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Idea: Try to play it safe. Assume that the other player tries to harm you as much as he can.
Def.: Let $G$ be a TSC; $x^* \in A_1$ is called a maximizer for player 1 if:

$$\min_{y \in A_2} u_1(x^*, y) \geq \min_{y \in A_2} u_1(x, y)$$

for all $x \in A_1$.

Similarly for player 2.
U.a. \( u_i(b^+_i, b^-_i) \geq u_i(b^+_i, b^-_i) \) for all \( b^-_i \).

And \( u_i(b^+_i, b^-_i) > u_i(b^+_i, b^-_i) \) for some \( b^-_i \).