Sealed-Bid Auctions

An object has to be assigned to one player, say \( i_1, \ldots, i_3 \), in exchange for a payment. For each player \( i \), \( v_i \) is the valuation of player \( i \) of the object. (e.g., we assume that \( v_1 > v_2 > v_3 > \ldots > v_n \).)

Mechanism: Players simultaneously give their bids \( b_1, b_2, \ldots, b_n \geq 0 \). The object is given to the bidder \( i \) with maximal bid \( b_i \). Break ties by valuation order, i.e., if \( b_i = b_j \) and \( v_i \leq v_j \), then \( i \) will win. If \( i < j \).

For first-price auction:

\[
U_i(b) = \begin{cases} 
0, & \text{if player } i \text{ does not win} \\
 v_i - \max b_{-i}, & \text{otherwise.} 
\end{cases}
\]

Example: Three bidders 1, 2, 3.

\( v_1 = 100, \quad v_2 = 80, \quad v_3 = 53 \)

\( b_1 = 90, \quad b_2 = 85, \quad b_3 = 45 \)

Bidder 3 wins both types of auctions.

First-price auction: \( U_3(b) = v_3 - b_3 = 100 - 45 = 55 \).

Second-price auction: \( U_3(b) = v_3 - b_3 = 100 - 45 = 55 \).

First-price auction: The payment by the winner is the highest bid.

Second-price auction: The payment by the winner is the highest bid of non-winning bidders.

Formula:

\[
N = \{1, \ldots, n\} \\
A_i = \{ b_i \mid b_i \in \mathbb{R}_+ \} \\
U_i(b) = \begin{cases} 
0, & \text{if player } i \text{ does not win} \\
 v_i - b_i, & \text{otherwise.} 
\end{cases}
\]

Proposition: In a second-price auction, bidding your own valuation, \( b_i \), is a weakly dominant strategy.

Proof: 1) Regardless of what the other bidders do, \( b_i \) is always a best \( b_i \). strategy.

Case I) \( i \) wins: \( i \) loses to pay \( \max b_{-i} \leq v_i \), which means that \( U_i(b_{-i}, b_i^+) = 0 \).

Case I.1) \( i \) decreases his bid: does not help. (\( b_i \) will still win at the same payment, or will lose at a lower payment.)

Case I.2) \( i \) increases his bid: \( i \) still wins, pays the same amount as before.
Case II.1: \( i \) guesses his bid:
- \( u_x(b^*, b^*_i) = 0 \).

Case II.2: \( i \) increases his bid:
- If \( i \) still loses, \( u_x = 0 \);
- if \( i \) becomes winner, \( i \) pays more than the object is worth to \( u_x = \text{negative utility} \).

2) \( b^*_i \) is strictly better than any other strategy and some opponent profile \( b^{-i} \).

Let \( b^*_i \) be some strategy \( \neq b^*_i \).

Case I) \( b^*_i < b^*_i \). Now let us consider \( b^{-i} \) with \( b^*_i > \max b^{-i} > b^*_i \). With \( b^*_i \), we do not win any more, i.e., we have \( u_x(b^{-i}, b^*_i) = 0 \), whereas \( u_x(b^{-i}, b^*_i) > 0 \).

Case II) \( b^*_i > b^*_i \). Consider \( b^{-i} > \max b^{-i} > b^*_i \).

Here \( u_x(b^{-i}, b^*_i) < 0 \), but \( u_x(b^{-i}, b^*_i) > 0 \).

Remark: A profile of weakly dominant strategies is a NE, because for no player there is an incentive to deviate to a different action.

Result: There is only NE, as seen.
**Two-Sum Games and NE**

**Def.** A two-sum game (2SG) is a 2-player strategic game

\[ G = \{A, B\}, (A_i \in A, (u_i)_{i \in A}\} \]

such that for all profiles \( a \in A \):

\[ u_a(a) + u_b(a) = 0. \]

Remark: Can be generalized to constant-sum games, where the utilities sum up to a constant \( c \).

**Ex.**

\[
\begin{array}{cccc|c}
\text{L} & \text{C} & \text{R} & \\
\hline
\text{T} & 8, -8 & 3, -3 & -6, 6 \\
\text{H} & 2, -2 & -1, 1 & 3, -7 \\
\text{B} & -5, 6 & 4, -4 & 8, -8 \\
\end{array}
\]

\[ \text{Max} \quad -6 \]

\[ \text{Min} \quad \text{Max} \quad \text{Min} \]

\[ \text{Idea: Try to play it safe. Assume that the other player tries to harm you as much as he can.} \]

**Def.** Let \( G \) be a 2SG; \( x^* \in A_1 \) is called a maximin for player 1 if:

\[ \min_{y \in A_2} u_a(x^*, y) \geq \min_{y \in A_2} u_a(x, y) \]

for all \( x \in A_1 \).

Similarly for player 2.

\[ u.a.: \quad u_a(b^*_i, b_i) \geq u_a(b^*_i, b_i) \quad \text{for all } b_i \]

and \( u_a(b^*_i, b_i) > u_a(b^*_i, b_i) \) for some \( b_i \).