Recall: A strategy $a_i^* \in \Theta_i$ is __strictly__ dominated by strategy $a_i^{+} \in \Theta_i$ iff there exists $a_{-i} \in \Theta_{-i}$:

$$u_i(a_i^+, a_{-i}) > u_i(a_i^*, a_{-i}).$$

Strategy $a_i^* \in \Theta_i$ is __weakly__ dominated by $a_i^{+} \in \Theta_i$ iff for all $a_{-i} \in \Theta_{-i}$:

$$u_i(a_i^+, a_{-i}) \geq u_i(a_i^*, a_{-i}).$$

and for some $a_{-i} \in \Theta_{-i}$:

$$u_i(a_i^+, a_{-i}) > u_i(a_i^*, a_{-i}).$$
**Def. (Nash equilibrium).**

A Nash equilibrium (NE) of a strategic game $G$ is a strategy profile $a^* \in A$ such that for all $i \in N$:

$$u_i(a^*) \geq u_i(a_{-i}, a_i) \quad \text{for } a_{-i} \in A_{-i}.$$

**Ex.:**

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>2, 2</td>
<td>0, 0</td>
</tr>
<tr>
<td>B</td>
<td>0, 0</td>
<td>1, 1</td>
</tr>
</tbody>
</table>

$u_1(A, A) = 2$

$(A, A)$ NE.
Ex. 2: Prisoner's Dilemma

```
   c   d
  +---+---+
  | 1,4| 3,3|
  +---+---+
  | 4,0| 3,3|
  +---+---+
```

1 - Cooperate
2 - Defect
## Ex. 3: Matching Pennies

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>H: 1, -1</td>
<td>T: -1, 1</td>
</tr>
<tr>
<td>T</td>
<td>H: -1, 1</td>
<td>T: 1, -1</td>
</tr>
</tbody>
</table>
Alt. Def. of NE:

\[ B_i(a_{-i}) = \{ a_i \in A_i \mid u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}) \text{ for } a'_i \in A_i \} \]

\[ B_i : A_{-i} \to 2^{A_i} \] is called best-response function.

A NE \( a^* \) is a profile \( a^* \) s.t.

\[ a^*_i \in B_i(a^*) \text{ for } i \in N. \]

I.e., \( a^* \in B(a^*) = \bigsqcap_{i=1}^n B_i(a^*_i) \).
Iterative elimination of weakly dominated strategies:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>M</td>
<td>2,1</td>
<td>1,1</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

NE: (T, L), (M, L), (M, R), (B, R)

1. dom. by M
2. dom. by L
Iterative Elimination of strictly dominated strategies.

Lemma: Let $G$ be a finite strategic game and $G'$ be the game resulting from eliminating one strictly dominated strategy from $G$. Then the NEs of $G$ are exactly the NEs of $G'$. 
Proof: Let $a_i'$ be the eliminated strategy. Then exist $a_i^+$ s.t. for $a_i, a_i' \in \Theta_i$:

$$u_i(a_{-i}, a_i') < u_i(a_{-i}, a_i^+) \quad (1)$$

\[ \Rightarrow \] : Let $a_i^*$ be a NE of $G$. Then

$$u_i(a_{-i}^*, a_i^*) > u_i(a_{-i}^*, a_i^+) \quad \text{for } a_i^+ \in \Theta_i$$

$$\Rightarrow \quad u_i(a_{-i}, a_i^*) > u_i(a_{-i}, a_i^+) > u_i(a_{-i}^*, a_i^+) \quad \text{(1)}$$

$$\Rightarrow \quad a_i^* \neq a_i' \Rightarrow \text{NE strategy was not eliminated}$$

$$\Rightarrow \quad a_i^* \text{ sh.D NE in } G'.$$
Let \( a^* \) be a NE in \( G' \).

For players \( j \neq i : a^*_j \in B_j^i (a^*_j) = B_j^i (a^-_j) \).
(no strategy of player \( j \) was eliminated.)

For player \( i : u_i (a^*_i, a^*_j) \geq u_i (a^*_i, a^+_i) \)

\[ u_i (a^*_i, a^+_i) > u_i (a^*_i, a^-_i) \tag{1} \]

\[ \implies a^*_i \text{ is not a best response to } a^-_i \text{ then } a^*_i \text{ (in } G) \]

\[ \implies a^*_i \in B_i^i (a^*_i) \implies a^* \text{ also NE in } G. \]
Corollary: If TEDS with strict dominance results in a unique strategy profile $a^*$, then $a^*$ is the unique NE of the original game.

Proof: Induction on previous Lemma.

Remark: TEDS wih strict dominance does not depend on elimination order.