Multiagent Systems

10. Coalition Formation

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Motivation

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Remember the prisoner's dilemma with the following **payoff matrix**:

		Player 2	
		C	D
Player 1	C	$2, 2$	$0, 3$
	D	$3, 0$	$1, 1$

In games like this one cooperation is prevented, because:

- Binding agreements are not possible
- Utility is given directly to individuals as the result of individual action

How about real world situations?

Prisoner's dilemma & the real world

Theoretical problems:

- **Binding agreements** are not possible
- Utility is given directly to **individuals** as the result of individual action

Real world situation:

- **Contracts** can form binding agreements
- Utility is given to **organizations**/groups of people and not to individuals

Under these circumstances cooperation becomes both possible and rational.

$\Rightarrow$ **Cooperative game theory** asks which contracts are meaningful solutions among self-interested agents.

Terminology

Terminology I

Setting:

- $Ag = \{1, \ldots, n\}$ agents (finite, typically $n > 2$)
- Any subset C of Ag is called a **coalition**
- $C = Ag$ is the **grand coalition**
- A **cooperative game** is a pair $\mathcal{G} = \langle Ag, \nu \rangle$
- $\nu : \mathbf{2}^{Ag} \to \mathbb{R}$ is the **characteristic function** of the game
- $\nu(C)$ is the maximum utility C can achieve, regardless of the remaining agents' behaviors (outside of coalition C)
- A coalition with only one agent is a **singleton coalition**

Finally: **individual actions**, **utilities**, and the origin of ν do not matter, i.e. they are assumed to be given.

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Example:

- A game with $Ag = \{1, 2\}$
- Singleton coalitions $\nu(\{1\}) = 5$ and $\nu(\{2\}) = 5$
- Grand coalition $\nu(\{1, 2\}) = 20$

Terminology II

A **simple coalition game**:

- value of any coalition is either 0 ('loosing') or 1 ('winning')
- **voting systems** can be understood in terms of simple games

General questions now:

1. Which coalitions might be formed by rational agents?
2. How should payoff be reasonably divided between members of a coalition?

⇒ Just as non-cooperative games had solution concepts (Nash-equilibria, . . .), cooperative games have theirs as well (Shapley value, . . .).

Basics

Three Stages of Cooperative Action

The **cooperation lifecycle** (Sandholm et al., 1999):

- Coalition structure generation:
  - Asking which coalitions will form, concerned with **stability**
  - For example, a productive agent has the incentive to defect from a coalition with a lazy agent
  - Necessary but not sufficient condition for establishment of a coalition
- Solving the optimization problem of each coalition:
  - Decide on collective plans
  - Maximize the **collective utility** of the coalition
- Dividing the value of the solution of each coalition:
  - Concerned with **fairness** of contract
  - How much an agent should receive based on her contribution

Outcome and Objections

Question: *Which coalitions are stable?*

- An **outcome** $x = \langle x_1, \ldots, x_k \rangle$ for a coalition C in game $\langle Ag, \nu \rangle$ is a distribution of C's utility to members of C
- Outcomes must be **feasible** (don't overspend) and efficient don't underspend) $\Rightarrow \sum_{i \in C} x_i = \nu(C)$
- Example:
  - $Ag = \{1, 2\}$, $\nu(\{1\}) = 5$, $\nu(\{2\}) = 5$, and $\nu(\{1, 2\}) = 20$
  - Possible outcomes for $C_{grand} = \{1, 2\}$ are $\langle 20, 0 \rangle$, $\langle 19, 1 \rangle$, $\ldots$, $\langle 1, 19 \rangle$, $\langle 0, 20 \rangle$
- C (e.g. a singleton coalition) **objects** to an outcome of a **grand coalition** (e.g. $\langle 1, 19 \rangle$), if there is some outcome for C (e.g. $\nu(\{1\}) = 5$) in which all members of C are strictly better off

Formally: $C \subseteq Ag$ object to $x = \langle x_1, \ldots, x_n \rangle$ for the grand coalition, iff there exists some outcome $x' = \langle x'_1, \ldots, x'_k \rangle$ for C, such that $x'_i > x_i$ for all $i \in C$

The core

Answering the question "Is the grand coalition stable?" is the same as asking:

Is the core non-empty?

The core

The **core** of a coalition game is the set of outcomes for the grand coalition to which nobody has an objection.

Non-empty core $\Rightarrow$ there exists some way that the grand coalition can cooperate and distribute the resulting utility such that no (sub-)coalition could do better by defecting.

Previous example?

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Previous example?

Core contains all outcomes between $\langle 15, 5\rangle$ and $\langle 5, 15\rangle$ inclusive

The core: problems

Despite the usefulness of the concept of the core, some problems arise:

- Sometimes the **core is empty** and to detect this **all possible coalitions need to be enumerated** $\Rightarrow$ with n agents, 2^{n-1} subsets / coalitions need to be checked!
- **Fairness** is not considered, e.g. $\langle 5, 15\rangle \in core$, but all surplus goes to one agent alone

Solution to second problem is considered next.

Shapley value

Shapley value (preliminaries)

Idea: To eliminate unfair outcomes, try to divide surplus according to each agent's contribution

Define marginal contribution of i to C:

Marginal contribution

The marginal contribution $\mu_i(C)$ of agent i to coalition C is defined as: $\mu_i(C) = \nu(C \cup \{i\}) - \nu(C)$

Axioms any fair distribution should satisfy:

- **Symmetry**: if two agents contribute the same, then they should receive same payoff (they are interchangeable)
- **Dummy player**: agents not adding any value to any coalition should receive what they earn on their own
- **Additivity**: if two games are combined, then the value a player gets should equal the sum of the values it receives in the individual games

Shapley value

Shapley value

The Shapley value sh_i for agent i is defined as:

$$sh_i = \frac{1}{|Ag|!} \sum_{o \in \prod(Ag)} \mu_i(C_i(o))$$

- $\prod(Ag)$ denotes the set of all possible orderings, i.e. permutations, for example, with $Ag = \{1, 2, 3\}$: $\prod(Ag) = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), \ldots\})$
- $C_i(o)$ denotes the set containing only those agents that appear before agent i in o, for example, with $o = \{3, 1, 2)$: $C_3(o) = \emptyset$ and $C_2(o) = \{1, 3\}$
- Requires that $\nu(\emptyset) = 0$ and $\nu(C \cup C') \geq \nu(C) + \nu(C')$ if $C \cap C' = \emptyset$ (i.e. ν must be superadditive)

Shapley value: examples

Examples for calculations of the Shapley value:

1. Consider $\nu(\{1\}) = 5$, $\nu(\{2\}) = 5$, and $\nu(\{1,2\}) = 20$:
   - Intuition says to allocate 10 to each agent
   - $\mu_1(\emptyset) = 5$, $\mu_2(\emptyset) = 5$, $\mu_1(\{2\}) = 15$, $\mu_2(\{1\}) = 15$
     $\Rightarrow sh_1 = sh_2 = (5 + 15)/2 = 10$ (same as intuition)
2. Consider $\nu(\{1\}) = 5$, $\nu(\{2\}) = 10$, and $\nu(\{1,2\}) = 20$:
   - $\mu_1(\emptyset) = 5$, $\mu_2(\emptyset) = 10$,
     $\mu_1(\{2\}) = \nu(\{1,2\}) - \nu(\{2\}) = 20 - 10 = 10$,
     $\mu_2(\{1\}) = 20 - 5 = 15$
     $\Rightarrow sh_1 = (5 + 10)/2 = 7.5$, $sh_2 = (10 + 15)/2 = 12.5$
   - Agent 2 contributes more than agent 1
     $\Rightarrow$ receives higher payoff!

Shapley value: a dummy player example

Finally, consider $Ag = \{1,2,3\}$, with $\nu(\{1\}) = 5$, $\nu(\{2\}) = 5$, $\nu(\{3\}) = 5$, $\nu(\{1,2\}) = 10$, $\nu(\{1,3\}) = 10$, $\nu(\{2,3\}) = 20$, and $\nu(\{1,2,3\}) = 25$:

- We have $\mu_1(\emptyset) = 5$, $\mu_2(\emptyset) = 5$, $\mu_3(\emptyset) = 5$, $\mu_1(\{2\}) = 5$, $\mu_1(\{3\}) = 5$, $\mu_1(\{2,3\}) = 5$, $\mu_2(\{1\}) = 5$, $\mu_2(\{3\}) = 15$, $\mu_2(\{1,3\}) = 15$, $\mu_3(\{2\}) = 15$, $\mu_3(\{1,2\}) = 15$.
- Agent 1 is a **dummy player** and its share should be $sh_1 = 5$ (dummy player axiom)
- $sh_2 = (5+5+15+15)/4 = 10$ and similarly $sh_3 = 10$.

Important: The Shapley value is the **only value** that satisfies the fairness axioms

Representation

Computational and representational issues

Consider a **naïve representation** of a coalition game:

```
1, 2, 3
1 = 5
2 = 5
3 = 5
1, 2 = 10
1, 3 = 10
2, 3 = 20
1, 2, 3 = 25
```

This is **infeasible**, because it is **exponential** in the size of Ag!

$\Rightarrow$ **succinct** representation needed:

- Modular representations exploit Shapley's axioms directly
- Basic idea: divide the game into smaller games and exploit additivity axiom

Modular representations

Two modular representations will be discussed:

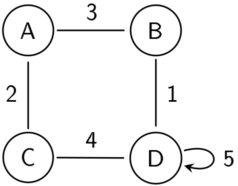
1. Induced subgraphs: a succinct, but incomplete representation
2. Marginal contribution nets: generalization of induced subgraphs, complete, but in worst case not succinct

Induced subgraphs

Idea: define characteristic function $\nu(C)$ by an undirected weighted graph

- Value of a coalition $C \subseteq Ag : \nu(C) = \sum_{\{i,j\} \subseteq C} w_{i,j}$

Example:



$$
\begin{aligned}
\nu(\{A,B,C\}) &= 3+2=5 \\
\nu(\{D\}) &= 5 \\
\nu(\{B,D\}) &= 1+5=6 \\
\nu(\{A,C\}) &= 2
\end{aligned}
$$

- Not a complete representation
- But easy to compute the Shapley value for a given player in polynomial time: $sh_i = \frac{1}{2} \sum_{j \neq i} w_{i,j}$

$\Rightarrow$ Checking emptiness of the core is NP-complete, and membership to the core is co-NP-complete

Marginal Contribution Nets I

Idea: represent characteristic function as a **set of rules**

pattern $\rightarrow$ **value**

1. Structure of the rules:
   - **pattern** is conjunction of agents, e.g. $1 \wedge 3$
   - $1 \wedge 3$ would apply to $\{1, 3\}$ and $\{1, 3, 5\}$, but not to $\{1\}$ or $\{8, 12\}$
   - $C \models \phi$: the rule $\phi \rightarrow x$ applies to coalition C
   - $rs_C = \{\phi \rightarrow x \in rs \mid C \models \phi\}$: the rules that apply to C
2. The characteristic function associated with the ruleset rs:

$$\nu_{rs}(C) = \sum_{\phi \rightarrow x \in rs_C} x$$

Marginal Contribution Nets II

Example:

- $rs_1 = \{a \wedge b \rightarrow 5, b \rightarrow 2\}$
- $\nu_{rs_1}(\{a\}) = 0$, $\nu_{rs_1}(\{b\})) = 2$, and $\nu_{rs_1}(\{a, b\})) = 7$

Extension:

- Allow negation in rules indicating the absence of agents instead of their presence
- Example: with $rs_2 = \{a \wedge b \rightarrow 5, b \rightarrow 2, c \rightarrow 4, b \wedge \neg c \rightarrow -2\}$ we have $\nu_{rs_2}(\{b\}) = 0$ (2nd and 4th rule), and $\nu_{rs_2}(\{b, c\}) = 6$ (2nd and 3rd rule)

General properties:

- Shapley value can be computed in polynomial time
- Complete representation, but not necessarily succinct

Representations for Simple Games

Remember: A coalition game is **simple**, if the value of any coalition is either zero (losing) or one (winning).

- Simple games model **yes/no** voting systems
- $Y = \langle Ag, W \rangle$, where $W \subseteq \mathbf{2}^{Ag}$ is the set of winning coalitions
- If $C \in W$, coalition C would be able to determine the outcome, 'yes' or 'no'

Important conditions:

- Non-triviality: $\emptyset \subset W \subset \mathbf{2}^{Ag}$
- Monotonicity: if $C_1 \subseteq C_2$ and $C_1 \in W$ then $C_2 \in W$
- Zero-sum: if $C \in W$ then $Ag \setminus C \notin W$
- Empty coalition loses: $\emptyset \notin W$
- Grand coalition wins: $Ag \in W$

Important: **Naïve representation** is **exponential** in the number of agents

Weighted Voting Games

Weighted voting games are an extension of simple games:

- For each agent $i \in Ag$ define a **weight** w_i
- Define an **overall quota** q
- A **coalition is winning** if the sum of their weights **exceeds the quota**:

$$\nu(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

Example: **Simple majority voting**, $w_i = 1$ and $q = \frac{\lceil |Ag|+1 \rceil}{2}$

- Succinct (but incomplete) representation: $\langle q; w_1, \ldots, w_n \rangle$

Shapley-Shubic power index

The **Shapley-Shubic power in index** is the Shapley value in yes/no games:

- Measures the power of the voter in this case
- Computation is NP-hard, no reasonable polynomial time approximation
- Checking emptiness of the core can be done in polynomial time (veto player)

It has counter-intuitive properties:

- In the weighted voting game $\langle 100; 99, 99, 1 \rangle$ all three voters have **the same power** $(\frac{1}{3})$
- Player with non-zero weight might nevertheless have no power, e.g., in $\langle 10; 6, 4, 2 \rangle$ third player is a **dummy player**
- But, by adding one player $\langle 10; 6, 4, 2, 8 \rangle$ third player's power increases

k-weighted Voting Games

Extension of weighted voting games:
⇒ **k-weighted voting games**

- complete representation (in contrast to weighted voting games)
- overall game: "conjunction" k of k different weighted voting games
- Winning coalition: the one that wins in **all component games**

Relation to **simple coalition games** (Wooldridge, p. 285):

> "Every **simple game** can be represented by a **k-weighted voting game** in which k is **at most exponential** in the number of players."

Real world relevance: the **voting system of the enlarged European Union** is a three-weighted voting game

Summary

Summary

What we have learned today:

- Coalition formation
- The core of a coalition game
- The Shapley value
- Different representations for different types of games
  - General coalition games: induced subgraphs & marginal contribution nets
  - Simple games: (k-)weighted voting games
- The Shapley-Shubic power index of simple games

Next (on Friday!):
Coalition Games with Goals & Coalition Structure Formation

Acknowledgments

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- Dr. Michael Rovatsos, The University of Edinburgh
  `http://www.inf.ed.ac.uk/teaching/courses/abs/abs-timetable.html`
- Michael Wooldridge: **An Introduction to MultiAgent Systems**, John Wiley & Sons, 2nd edition 2009.