10.1 Motivation

Remember the prisoner’s dilemma with the following payoff matrix:

\[
\begin{array}{c|cc}
    & \text{Player 1} & \text{Player 2} \\
\hline
    \text{C} & 2,2 & 0,3 \\
    \text{D} & 3,0 & 1,1 \\
\end{array}
\]

In games like this one cooperation is prevented, because:

▶ Binding agreements are not possible
▶ Utility is given directly to individuals as the result of individual action

How about real world situations?
Prisoner’s dilemma & the real world

Theoretical problems:
- **Binding agreements** are not possible
- Utility is given directly to **individuals** as the result of individual action

Real world situation:
- **Contracts** can form binding agreements
- Utility is given to **organizations/groups** of people and not to individuals

Under these circumstances cooperation becomes both possible and rational.
⇒ **Cooperative game theory** asks which contracts are meaningful solutions among self-interested agents.

**Terminology I**

**Setting:**
- \( Ag = \{1, \ldots, n\} \) agents (finite, typically \( n > 2 \))
- Any subset \( C \) of \( Ag \) is called a **coalition**
- \( C = Ag \) is the **grand coalition**
- A **cooperative game** is a pair \( G = (Ag, \nu) \)
- \( \nu : 2^{Ag} \to \mathbb{R} \) is the **characteristic function** of the game
- \( \nu(C) \) is the maximum utility \( C \) can achieve, regardless of the remaining agents' behaviors (outside of coalition \( C \))
- A coalition with only one agent is a **singleton coalition**

Finally: **individual actions**, **utilities**, and the origin of \( \nu \) do not matter, i.e. they are assumed to be given.

Example:
- A game with \( Ag = \{1, 2\} \)
- Singleton coalitions \( \nu(\{1\}) = 5 \) and \( \nu(\{2\}) = 5 \)
- Grand coalition \( \nu(\{1, 2\}) = 20 \)

**Terminology II**

A **simple coalition game**:
- value of any coalition is either 0 (‘loosing’) or 1 (‘winning’)
- **voting systems** can be understood in terms of simple games

General questions now:
1. Which coalitions might be formed by rational agents?
2. How should payoff be reasonably divided between members of a coalition?

⇒ Just as non-cooperative games had solution concepts (Nash-equilibria, ...), cooperative games have theirs as well (Shapley value, ...).
### Three Stages of Cooperative Action

The cooperation lifecycle (Sandholm et al., 1999):

- **Coalition structure generation:**
  - Asking which coalitions will form, concerned with **stability**
  - For example, a productive agent has the incentive to defect from a coalition with a lazy agent
  - Necessary but not sufficient condition for establishment of a coalition
- Solving the optimization problem of each coalition:
  - Decide on collective plans
  - Maximize the **collective utility** of the coalition
- Dividing the value of the solution of each coalition:
  - Concerned with **fairness** of contract
  - How much an agent should receive based on her contribution

### Outcome and Objections

**Question:** Which coalitions are stable?

- An **outcome** \( x = (x_1, \ldots, x_k) \) for a coalition \( C \) in game \( (Ag, \nu) \) is a distribution of \( C \)'s utility to members of \( C \)
- Outcomes must be **feasible** (don’t overspend) and efficient don’t underspend \( \Rightarrow \sum_{i \in C} x_i = \nu(C) \)

**Example:**

- \( Ag = \{1, 2\}, \nu(\{1\}) = 5, \nu(\{2\}) = 5 \) and \( \nu(\{1, 2\}) = 20 \)
- Possible outcomes for \( C_{\text{grand}} = \{1, 2\} \) are \( \langle 20, 0 \rangle, \langle 19, 1 \rangle, \ldots, \langle 1, 19 \rangle, \langle 0, 20 \rangle \)

- **Core** (e.g. a singleton coalition) objects to an outcome of a **grand coalition** (e.g. \( \langle 1, 19 \rangle \)), if there is some outcome for \( C \) (e.g. \( \nu(\{1\}) = 5 \)) in which all members of \( C \) are strictly better off

Formally: \( C \subseteq Ag \) object to \( x = (x_1, \ldots, x_n) \) for the grand coalition, iff there exists some outcome \( x' = (x'_1, \ldots, x'_n) \) for \( C \), such that \( x'_i > x_i \) for all \( i \in C \)

### The Core

Answering the question “Is the grand coalition stable?” is the same as asking:

**Is the core non-empty?**

**The core**

The core of a coalition game is the set of outcomes for the grand coalition to which nobody has an objection.

Non-empty core \( \Rightarrow \) there exists some way that the grand coalition can cooperate and distribute the resulting utility such that no (sub-)coalition could do better by defecting.

Previous example?

Core contains all outcomes between \( \langle 15, 5 \rangle \) and \( \langle 5, 15 \rangle \) inclusive
The core: problems

Despite the usefulness of the concept of the core, some problems arise:

- Sometimes the core is empty and to detect this all possible coalitions need to be enumerated ⇒ with n agents, \(2^n - 1\) subsets / coalitions need to be checked!
- Fairness is not considered, e.g. \((5, 15) \in \text{core}\), but all surplus goes to one agent alone

Solution to second problem is considered next.

10.4 Shapley value

Shapley value (preliminaries)

Idea: To eliminate unfair outcomes, try to divide surplus according to each agent’s contribution

Define marginal contribution of \(i\) to \(C\):

Marginal contribution

The marginal contribution \(\mu_i(C)\) of agent \(i\) to coalition \(C\) is defined as:

\[
\mu_i(C) = \nu(C \cup \{i\}) - \nu(C)
\]

Axioms any fair distribution should satisfy:

- **Symmetry**: if two agents contribute the same, then they should receive same payoff (they are interchangeable)
- **Dummy player**: agents not adding any value to any coalition should receive what they earn on their own
- **Additivity**: if two games are combined, then the value a player gets should equal the sum of the values it receives in the individual games

\[
\nu(\emptyset) = 0 \text{ and } \nu(C \cup C') \geq \nu(C) + \nu(C') \text{ if } C \cap C' = \emptyset
\]

\(\Pi(Ag)\) denotes the set of all possible orderings, i.e. permutations, for example, with \(Ag = \{1, 2, 3\}\):

\[
\Pi(Ag) = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), \ldots\}
\]

\(C_i(o)\) denotes the set containing only those agents that appear before agent \(i\) in \(o\), for example, with \(o = \{3, 1, 2\} \colon C_3(o) = \emptyset\) and \(C_2(o) = \{1, 3\}\)

The Shapley value \(sh_i\) for agent \(i\) is defined as:

\[
sh_i = \frac{1}{|Ag|!} \sum_{o \in \Pi(Ag)} \mu_i(C_i(o))
\]
Shapley value: examples

Examples for calculations of the Shapley value:

1. Consider $\nu(\{1\}) = 5$, $\nu(\{2\}) = 5$, and $\nu(\{1, 2\}) = 20$:
   - Intuition says to allocate 10 to each agent
   - $\mu_1(\emptyset) = 5$, $\mu_2(\emptyset) = 5$, $\mu_1(\{2\}) = 15$, $\mu_2(\{1\}) = 15$
   - $s_{h1} = s_{h2} = (5 + 15)/2 = 10$ (same as intuition)

2. Consider $\nu(\{1\}) = 5$, $\nu(\{2\}) = 10$, and $\nu(\{1, 2\}) = 20$:
   - $\mu_1(\emptyset) = 5$, $\mu_2(\emptyset) = 10$, $\mu_1(\{2\}) = 15$, $\mu_2(\{1\}) = 15$
   - $s_{h1} = (5 + 10)/2 = 7.5$, $s_{h2} = (10 + 15)/2 = 12.5$
   - Agent 2 contributes more than agent 1
   - $\Rightarrow$ receives higher payoff!

Finally, consider $Ag = \{1, 2, 3\}$, with $\nu(\{1\}) = 5$, $\nu(\{2\}) = 5$, $\nu(\{3\}) = 5$, $\nu(\{1, 2\}) = 10$, $\nu(\{1, 3\}) = 10$, $\nu(\{2, 3\}) = 20$, and $\nu(\{1, 2, 3\}) = 25$:

- We have $\mu_1(\emptyset) = 5$, $\mu_2(\emptyset) = 5$, $\mu_1(\{2\}) = 5$, $\mu_1(\{3\}) = 5$, $\mu_1(\{1\}) = 5$, $\mu_2(\{1, 3\}) = 15$, $\mu_2(\{1\}) = 15$, $\mu_3(\{1, 2\}) = 15$.
- Agent 1 is a dummy player and its share should be $s_{h1} = 5$ (dummy player axiom)
- $s_{h2} = (5 + 5 + 15 + 15)/4 = 10$ and similarly $s_{h3} = 10$.

Important: The Shapley value is the only value that satisfies the fairness axioms

### 10.5 Representation

- Induced subgraphs
- Marginal Contribution Nets
- Simple games

Computational and representational issues

Consider a naïve representation of a coalition game:

1, 2, 3
1 = 5
2 = 5
3 = 5
1, 2 = 10
1, 3 = 10
2, 3 = 20
1, 2, 3 = 25

This is infeasible, because it is exponential in the size of $Ag$!

$\Rightarrow$ succinct representation needed:

- Modular representations exploit Shapley’s axioms directly
- Basic idea: divide the game into smaller games and exploit additivity axiom
Modular representations

Two modular representations will be discussed:
1. Induced subgraphs: a succinct, but incomplete representation
2. Marginal contribution nets: generalization of induced subgraphs, complete, but in worst case not succinct

Induced subgraphs

Idea: define characteristic function \( \nu(C) \) by an undirected weighted graph

- Value of a coalition \( C \subseteq Ag : \nu(C) = \sum_{(i,j) \in C} w_{ij} \)

Example:

- For two agents, \( A \) and \( B \), the characteristic function values are:
  - \( \nu(\{A, B, C\}) = 3 + 2 = 5 \)
  - \( \nu(\{D\}) = 5 \)
  - \( \nu(\{B, D\}) = 1 + 5 = 6 \)
  - \( \nu(\{A, C\}) = 2 \)

- Not a complete representation
- But easy to compute the Shapley value for a given player in polynomial time:
  \[ sh_i = \frac{1}{|I|} \sum_{j \neq i} w_{ij} \]
- Checking emptiness of the core is NP-complete, and membership to the core is co-NP-complete

Marginal Contribution Nets

Idea: represent characteristic function as a set of rules

- \( \text{pattern} \rightarrow \text{value} \)

1. Structure of the rules:
   - \( \text{pattern} \) is conjunction of agents, e.g. \( 1 \land 3 \)
   - \( 1 \land 3 \) would apply to \( \{1,3\} \) and \( \{1,3,5\} \), but not to \( \{1\} \) or \( \{8,12\} \)
   - \( C \models \phi \): the rule \( \phi \rightarrow x \) applies to coalition \( C \)
   - \( rs_C = \{ \phi \rightarrow x \in rs \mid C \models \phi \} \): the rules that apply to \( C \)

2. The characteristic function associated with the ruleset \( rs \):

\[
\nu_{rs}(C) = \sum_{\phi \rightarrow x \in rs_C} x
\]

Example:

- \( rs_1 = \{ a \land b \rightarrow 5, b \rightarrow 2 \} \)
- \( \nu_{rs_1}(\{a\}) = 0, \nu_{rs_1}(\{b\}) = 2, \) and \( \nu_{rs_1}(\{a, b\}) = 7 \)

Extension:

- Allow negation in rules indicating the absence of agents instead of their presence
- Example: with \( rs_2 = \{ a \land b \rightarrow 5, b \rightarrow 2, c \rightarrow 4, b \land \neg c \rightarrow -2 \} \) we have \( \nu_{rs_2}(\{b\}) = 0 \) (2nd and 4th rule), and \( \nu_{rs_2}(\{b, c\}) = 6 \) (2nd and 3rd rule)

General properties:

- Shapley value can be computed in polynomial time
- Complete representation, but not necessarily succinct
Representations for Simple Games

Remember: A coalition game is simple, if the value of any coalition is either zero (losing) or one (winning).

- Simple games model yes/no voting systems
- \( Y = (\mathcal{A}, W) \), where \( W \subseteq 2^{\mathcal{A}} \) is the set of winning coalitions
- If \( C \in W \), coalition \( C \) would be able to determine the outcome, 'yes' or 'no'

Important conditions:
- Non-triviality: \( \emptyset \subset W \subset 2^{\mathcal{A}} \)
- Monotonicity: if \( C_1 \subseteq C_2 \) and \( C_1 \in W \) then \( C_2 \in W \)
- Zero-sum: if \( C \in W \) then \( \mathcal{A} \setminus C \notin W \)
- Empty coalition loses: \( \emptyset \notin W \)
- Grand coalition wins: \( \mathcal{A} \in W \)

Important: Naïve representation is exponential in the number of agents

Weighted Voting Games

Weighted voting games are an extension of simple games:
- For each agent \( i \in \mathcal{A} \) define a weight \( w_i \)
- Define an overall quota \( q \)
- A coalition is winning if the sum of their weights exceeds the quota:
  \[
  \nu(C) = \begin{cases} 
  1 & \text{if } \sum_{i \in C} w_i \geq q \\
  0 & \text{otherwise}
  \end{cases}
  \]

Example: Simple majority voting, \( w_i = 1 \) and \( q = \lceil |\mathcal{A}| + 1 \rceil / 2 \)
- Succinct (but incomplete) representation: \( (q; w_1, \ldots, w_n) \)

k-weighted Voting Games

Extension of weighted voting games:
- \( k \)-weight voting games
  - complete representation (in contrast to weighted voting games)
- overall game: “conjunction” \( k \) of \( k \) different weighted voting games
- Winning coalition: the one that wins in all component games

Relation to simple coalition games (Wooldridge, p. 285):

“Every simple game can be represented by a \( k \)-weighted voting game in which \( k \) is at most exponential in the number of players.”

Real world relevance: the voting system of the enlarged European Union is a three-weighted voting game


10.6 Summary

- Thanks

Summary

What we have learned today:

- Coalition formation
- The core of a coalition game
- The Shapley value
- Different representations for different types of games
  - General coalition games: induced subgraphs & marginal contribution nets
  - Simple games: (k-)weighted voting games
- The Shapley-Shubik power index of simple games

Next (on Friday!):
Coalition Games with Goals & Coalition Structure Formation

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- Dr. Michael Rovatsos, The University of Edinburgh
  http://www.inf.ed.ac.uk/teaching/courses/abs/abs-timetable.html
- Michael Wooldridge: An Introduction to MultiAgent Systems,