Procedural Remark

Course evaluation:
- Yay! It’s time for the course evaluation once again!
- You should have received emails with all the necessary details.
- Please complete the evaluation by July 27th, so we can discuss it on July 31st.
- Separate evaluations for BN and RM.

Motivation

- Preference relations≺contain no information about “by how much” one candidate is preferred.
- Idea: Use money to measure this.
- Use money also for transfers between players “for compensation”.

Setting

Formalization:
- Set of alternatives $A$.
- Set of $n$ players $I$.
- Valuation functions $v_i : A \rightarrow \mathbb{R}$ such that $v_i(a)$ denotes the value player $i$ assigns to alternative $a$.
- Payment functions specifying amount $m_i \in \mathbb{R}$ that player $i$ receives (or pays).
- Utility of player $i$: $u_i(a) = v_i(a) + m_i$. 
Second Price Auctions

Second price auctions:
- There are \( n \) players bidding for a single item.
- Player \( i \)'s private valuations of item: \( w_i \).
- Desired outcome: Player with highest private valuation wins bid.
- Players should reveal their valuations truthfully.
- Winner \( i \) pays price \( p^* \) and has utility \( w_i - p^* \).
- Non-winners pay nothing and have utility 0.

Formally:
- \( A = N \)
- \( v_i(a) = \begin{cases} w_i & \text{if } a = i \\ 0 & \text{else} \end{cases} \)
- What about payments? Say player \( i \) wins:
  - \( p^* = 0 \) (winner pays nothing): bad idea, players would manipulate and publicly declare values \( w'_i \gg w_i \).
  - \( p^* = w_i \) (winner pays his valuation): bad idea, players would manipulate and publicly declare values \( w'_i = w_i - \varepsilon \).
  - better: \( p^* = \max_{j \neq i} w_j \) (winner pays second highest bid).

Vickrey Auction

Definition (Vickrey Auction)
The winner of the Vickrey Auction (aka second price auction) is the player \( i \) with the highest declared value \( w_i \). He has to pay the second highest declared bid \( p^* = \max_{j \neq i} w_j \).

Proposition (Vickrey)
Let \( i \) be one of the players and \( w_i \) his valuation for the item, \( u_i \) his utility if he truthfully declares \( w_i \) as his valuation of the item, and \( u'_i \) his utility if he falsely declares \( w'_i \) as his valuation of the item. Then \( u_i \geq u'_i \).

Proof

Incentive Compatible Mechanisms

Idea: Generalization of Vickrey auctions.
- Preferences modeled as functions \( v_i : A \rightarrow \mathbb{R} \).
- Let \( V_i \) be the space of all such functions for player \( i \).
- Unlike for social choice functions: Not only decide about chosen alternative, but also about payments.
Mechanisms

Definition (Mechanism)
A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) consists of
- a social choice function \( f : V_1 \times \cdots \times V_n \to A \) and
- for each player \( i \), a payment function \( p_i : V_1 \times \cdots \times V_n \to \mathbb{R} \).

Definition (Incentive Compatibility)
A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) is called incentive compatible if for each player \( i = 1, \ldots, n \), for all preferences \( v_1 \in V_1, \ldots, v_n \in V_n \) and for each preference \( v'_i \in V_i \),
\[
v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}).
\]

VCG Mechanisms

Definition (Vickrey-Clarke-Groves mechanism)
A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) is called a Vickrey-Clarke-Groves mechanism (VCG mechanism) if
- \( f(v_1, \ldots, v_n) \in \arg\max_{a \in A} \sum_{i=1}^n v_i(a) \) for all \( v_1, \ldots, v_n \) and
- there are functions \( h_1, \ldots, h_n \) with \( h_i : V_{-i} \to \mathbb{R} \) such that \( p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n)) \) for all \( i = 1, \ldots, n \) and \( v_1, \ldots, v_n \).

Note: \( h_i(v_{-i}) \) independent of player \( i \)'s declared preference \( \Rightarrow h_i(v_{-i}) = c \) constant from player \( i \)'s perspective.

Utility of player \( i \) is \( v_i(f(v_1, \ldots, v_n)) + \sum_{j \neq i} v_j(f(v_1, \ldots, v_n)) - c = \sum_{j=1}^n v_j(f(v_1, \ldots, v_n)) - c = \) social welfare - \( c \).

VCG Mechanisms

Theorem (Vickrey-Clarke-Groves)
Every VCG mechanism is incentive compatible.

Proof
Let \( i, v_{-i}, v_i \) and \( v'_i \) be given. Show: Declaring true preference \( v_i \) dominates declaring false preference \( v'_i \).

Let \( a = f(v_i, v_{-i}) \) and \( a' = f(v'_i, v_{-i}) \).

Utility player \( i \) is
\[
\begin{cases}
  v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \text{if declaring } v_i \\
  v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \text{if declaring } v'_i
\end{cases}
\]

Alternative \( a = f(v_i, v_{-i}) \) maximizes social welfare
\[
\Rightarrow v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a').
\]
\[
\Rightarrow v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}).
\]
Clarke Pivot Rule

- So far: payment functions $p_i$ and functions $h_i$ unspecified.
- One possibility: $h_i(v_{-i}) = 0$ for all $h_i$ and $v_{-i}$.
- Drawback: Too much money distributed among players (more than necessary).
- Further requirements:
  - Players should pay at most as much as they value the outcome.
  - Players should only pay, never receive money.

Individual Rationality, Positive Transfers

Definition (individual rationality)
A mechanism is individually rational if all players always get a nonnegative utility, i.e., if for all $i = 1, \ldots, n$ and all $v_1, \ldots, v_n$,

$$v_i(f(v_1, \ldots, v_n)) - p_i(v_1, \ldots, v_n) \geq 0.$$  

Definition (positive transfers)
A mechanism has no positive transfers if no player is ever paid money, i.e., for all preferences $v_1, \ldots, v_n$,

$$p_i(v_1, \ldots, v_n) \geq 0.$$  

Clarke Pivot Function

Definition (Clarke pivot function)
The Clarke pivot function is the function

$$h_i(v_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b).$$

This leads to payment functions

$$p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$

for $a = f(v_1, \ldots, v_n)$.

Player $i$ pays the difference between what the other players could achieve without him and what they achieve with him.

Each player internalizes the externalities he causes.
Clarke Pivot Rule

Lemma (Clarke pivot rule)

A VCG mechanism with Clarke pivot functions has no positive transfers. If \( v_i(a) \geq 0 \) for all \( i = 1, \ldots, n \), \( v_i \in V_i \) and \( a \in A \), then the mechanism is also individually rational.

Proof

Let \( a = f(v_1, \ldots, v_n) \) be the alternative maximizing \( \sum_{j=1}^{n} v_j(a) \), and \( b \) the alternative maximizing \( \sum_{j \neq i} v_j(b) \).

Utility of player \( i \): \( u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \).

Payment function for \( i \): \( p_i(v_1, \ldots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \).

Since \( b \) maximizes \( \sum_{j \neq i} v_j(b) \): \( p_i(v_1, \ldots, v_n) \geq 0 \) (no positive transfers).

Vickery Auction as a VCG Mechanism

- \( A = N \). Valuations: \( w_i, v_a(a) = w_a, v_j(a) = 0, (i \neq a) \).
- \( a \) maximizes social welfare \( \sum_{i=1}^{n} v_i(a) \) iff \( a \) maximizes \( w_a \).
- Let \( a = f(v_1, \ldots, v_n) = \arg \max_{j \in A} w_j \) be the highest bidder.
- Payments: \( p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \).
- But \( \max_{b \in A} \sum_{j \neq i} v_j(b) = \max_{b \in A \setminus \{i\}} w_b \).
- Winner pays value of second highest bid:
  \[
  p_a(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq a} v_j(b) - \sum_{j \neq a} v_j(a) \\
  = \max_{b \in A \setminus \{a\}} w_b - 0 = \max_{b \in A \setminus \{a\}} w_b.
  \]
- Non-winners pay nothing: For \( i \neq a \),
  \[
  p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \\
  = \max_{b \in A \setminus \{i\}} w_b - w_a = w_a - w_a = 0.
  \]

Example: Bilateral Trade

- Seller \( s \) offers item he values with \( 0 \leq w_s \leq 1 \).
- Potential buyer \( b \) values item with \( 0 \leq w_b \leq 1 \).
- Alternatives \( A = \{\text{trade, no-trade}\} \).
- Valuations:
  \[
  v_s(\text{no-trade}) = 0, \quad v_s(\text{trade}) = -w_s, \quad v_b(\text{no-trade}) = 0, \quad v_b(\text{trade}) = w_b.
  \]
- VCG mechanism maximizes \( v_s(a) + v_b(a) \).
- We have
  \[
  v_s(\text{trade}) + v_b(\text{trade}) = w_b - w_s, \quad v_s(\text{no-trade}) + v_b(\text{no-trade}) = 0
  \]
  i.e., \( \text{trade} \) maximizes social welfare iff \( w_b \geq w_s \).
Example: Bilateral Trade (ctd.)

- **Requirement:** if no-trade is chosen, neither player pays anything:
  \[ p_s(v_s, v_b) = p_b(v_s, v_b) = 0. \]
- To that end, choose Clarke pivot function for buyer:
  \[ h_b(v_s) = \max_{a \in A} v_s(a). \]
- For seller: Modify Clarke pivot function by an additive constant and set
  \[ h_s(v_b) = \max_{a \in A} v_b(a) - w_b. \]

Because \( w_b \geq w_s \), the seller gets at least as much as the buyer pays, i.e., the mechanism subsidizes the trade.

Without subsidies, no incentive compatible bilateral trade possible.

**Note:** Buyer and seller can exploit the system by colluding.

Example: Bilateral Trade (ctd.)

- For alternative no-trade,
  \[ p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\text{no-trade}) \]
  \[ = w_b - w_b - 0 = 0 \quad \text{and} \]
  \[ p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\text{no-trade}) \]
  \[ = 0 - 0 = 0. \]

- For alternative trade,
  \[ p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\text{trade}) \]
  \[ = w_b - w_b - w_b = -w_b \quad \text{and} \]
  \[ p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\text{trade}) \]
  \[ = 0 + w_s = w_s. \]

Because \( w_b \geq w_s \), the seller gets at least as much as the buyer pays, i.e., the mechanism subsidizes the trade.

Without subsidies, no incentive compatible bilateral trade possible.

**Note:** Buyer and seller can exploit the system by colluding.

Example: Public Project

- **Project costs** \( C \) units.
- Each citizen \( i \) privately values the project at \( w_i \) units.
- Government will undertake project if \( \sum_i w_i > C \).
- **Alternatives:** \( A = \{ \text{project, no-project} \} \).
- **Valuations:**
  \[ v_G(\text{project}) = -C, \quad v_G(\text{no-project}) = 0, \]
  \[ v_i(\text{project}) = w_i, \quad v_i(\text{no-project}) = 0. \]

- **VCG mechanism with Clarke pivot rule:** for each citizen \( i \),
  \[ h_i(v_{-i}) = \max_{a \in A} \left( \sum_{j \neq i} v_j(a) + v_G(a) \right) \]
  \[ = \begin{cases} 
  \sum_{j \neq i} w_j - C, & \text{if } \sum_{j \neq i} w_j > C \\
  0, & \text{otherwise.}
  \end{cases} \]
Example: Public Project (ctd.)

- Citizen $i$ pivotal if $\sum_j w_j > C$ and $\sum_{j \neq i} w_j \leq C$.
- Payment function for citizen $i$:
  $$p_i(v_{1..}, V_G) = h_i(v_{-i}) - \left( \sum_{j \neq i} v_j(f(v_{1..}, V_G)) + V_G(f(v_{1..}, V_G)) \right)$$
- Case 1: Project undertaken, $i$ pivotal:
  $$p_i(v_{1..}, V_G) = 0 - \left( \sum_{j \neq i} w_j - C \right) = C - \sum_{j \neq i} w_j$$
- Case 2: Project undertaken, $i$ not pivotal:
  $$p_i(v_{1..}, V_G) = \left( \sum_{j \neq i} w_j - C \right) - \left( \sum_{j \neq i} w_j - C \right) = 0$$
- Case 3: Project not undertaken:
  $$p_i(v_{1..}, V_G) = 0$$

Example: Buying a Path in a Network

- Communication network modeled as $G = (V, E)$.
- Each link $e \in E$ owned by different player $e$.
- Each link $e \in E$ has cost $c_e$, if used.
- Objective: procure communication path from $s$ to $t$.
- Alternatives: $A = \{p | p \text{ path from } s \text{ to } t \}$.
- Valuations: $v_e(p) = -c_e$, if $e \in p$, and $v_e(p) = 0$, if $e \notin p$.
- Maximizing social welfare:
  minimize $\sum_{e \in p} c_e$ over all paths $p$ from $s$ to $t$.
- Example:

Example: Buying a Path in a Network (ctd.)

- For $G = (V, E)$ and $e \in E$ let $G \setminus e = (V, E \setminus \{e\})$.
- VCG mechanism with Clarke pivot function:
  $$h_e(v_{-e}) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_e'$$
  i.e., the cost of the cheapest path from $s$ to $t$ in $G \setminus e$.
  (Assume that $G$ is 2-connected, s.t. such $p'$ exists.)
- Payment functions: for chosen path $p = f(v_1, \ldots, v_n)$,
  $$p_e(v_1, \ldots, v_n) = h_e(v_{-e}) - \sum_{e' \neq e \in p} -c_{e'}.$$
Example: Buying a Path in a Network (ctd.)

- **Example:**

  
  
  \[
  \begin{align*}
  c_a &= 4 \\
  c_b &= 3 \\
  c_c &= 4 \\
  c_d &= 12 \\
  c_e &= 5 \\
  \end{align*}
  \]

  
  - Cost along $b$ and $e$: 8
  - Cost without $e$: 3
  - Cost of cheapest path without $e$: 15 (along $b$ and $d$)
  - Difference is payment: $-15 - (-3) = -12$
  - i.e., owner of arc $e$ gets paid 12 for using his arc.

- **Note:** Alternative path after deletion of $e$ does not necessarily differ from original path at only one position. Could be totally different.

Summary

- New preference model: with **money**.
- VCG mechanisms generalize **Vickrey auctions**.
- VCG mechanisms are **incentive compatible mechanisms** maximizing social welfare.
- With Clarke pivot rule: **even no positive transfers and individually rational** (if nonnegative valuations).
- Various application areas.