Social Choice Theory

- Aggregation of preferences of group members
- Voting and voting procedures
  - Elections
  - Committee decisions
  - European Song Contest

Def (Social Welfare and Social Choice Functions)

Let $A$ be a set of alternatives (candidates) and $L$ be a set of linear orders on $A$. For $n$ voters, $F: L^n \rightarrow L$ denotes a social welfare function and $f: L^n \rightarrow A$ a social choice function.

Notation: A linear order $\prec \in L$ is called a preference relation. For voter $i$, $\prec_i$. For example, $a \prec_i b$ means that voter $i$ prefers candidate $b$ over candidate $a$.

Example:
Assume there are voters: $1, 2, 3$
and three alternatives: $a, b, c$

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>a</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

Voting protocols

- Plurality: (also first-past-the-post)
  - Only top preferences are taken into account
  - Candidate with most top preferences wins
  - Drawback: Waste votes, winner might be preferred only by a minority.

- Plurality with runoff
  - First round: Two candidates with most top votes proceed to second round
  - Second round: Runoff
  - Drawback: Takes more time, tactical voting is possible.
Instant runoff voting (transferable votes):
- each voter submits a preference order
- identically candidates with the fewest top preferences are eliminated until
  one candidate remains.

Drawback: Compromise candidates might yet not elected.

Borda count:
- each voter submits his preferences order
  over m candidates
- if a candidate is in position j of a voter’s list, he gets m-j points from that voter
- points from all votes are added
- candidate with most points wins.

<table>
<thead>
<tr>
<th>23 voters, candidates: a, b, c, d, e</th>
</tr>
</thead>
<tbody>
<tr>
<td># votes</td>
</tr>
<tr>
<td>1st</td>
</tr>
<tr>
<td>2nd</td>
</tr>
<tr>
<td>3rd</td>
</tr>
<tr>
<td>4th</td>
</tr>
<tr>
<td>5th</td>
</tr>
</tbody>
</table>

Plurality voting: e

Condorced winner:
- each voter submits his preference order
- perform a pairwise comparison between candidates
- if one candidate wins all pairwise comparisons, he is the Condorced winner.

Drawback: Condorced winner does not always exist.

<table>
<thead>
<tr>
<th>23 voters, candidates: a, b, c, d, e</th>
</tr>
</thead>
<tbody>
<tr>
<td># votes</td>
</tr>
<tr>
<td>1st</td>
</tr>
<tr>
<td>2nd</td>
</tr>
<tr>
<td>3rd</td>
</tr>
<tr>
<td>4th</td>
</tr>
<tr>
<td>5th</td>
</tr>
</tbody>
</table>

Plurality voting with run-off
1st round: e and a
2nd round: 6e + 4x a + 2x a + 1xe = 14e
8x e + 1xe = 9xe

a is the winner.
23 voters, candidates: a, b, c, d, e

<table>
<thead>
<tr>
<th>Rank</th>
<th>Number</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>6</td>
<td>e</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>d</td>
</tr>
<tr>
<td>2nd</td>
<td>4</td>
<td>e</td>
<td>c</td>
<td>d</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>3rd</td>
<td>3</td>
<td>a</td>
<td>c</td>
<td>b</td>
<td>d</td>
<td>e</td>
</tr>
<tr>
<td>4th</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
</tr>
<tr>
<td>5th</td>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
</tr>
</tbody>
</table>

2x e
d

Instant runoff voting:
1st round eliminated: d
2nd round: b
3rd round: a
Now c has the majority

Borda count:
a: 8.0 + 6.4 + 4.1 + 3.4 + 1.2 + 1.0 = 33
b: 8.2 + 6.3 + 4.9 + 3.3 + 1.1 + 1.2 = 25
b: 8.4 + 6.2 + 4.9 + 3.3 + 1.1 + 1.2 = 25
c: 8.0 + 6.4 + 4.1 + 3.4 + 1.2 + 1.0 = 32
d: 8.2 + 6.3 + 4.9 + 3.3 + 1.1 + 1.2 = 25

23 voters, candidates: a, b, c, d, e

- Different winners for different voting protocols
- Which voting protocol prefer?
- Can be used strategically!
Condorcet Paradox

Example of votes 1, 2, 3 on candidates a, b, c:

\[
\begin{array}{ccc}
\ & a & b & c \\
\hline 
a & 2 & 1 & 1 \\
b & 1 & 2 & 1 \\
c & 1 & 1 & 2 \\
\end{array}
\]

\[a < c \quad c < b \quad b < c\]

Cyclic order

Def A Condorcet method returns a Condorcet winner if one exists. Otherwise it computes some winner.

One particular Condorcet method:

Schulze method

Relatively new (1997)

Already used by many: Debian, Ubuntu, Pirate Party.

Schulze Method:

Notation: \(d(x, y) = \text{number of pairwise comparisons won by } x \text{ over } y.\)

Def For candidates \(x\) and \(y\) there exists a path \(c_1, \ldots, c_n\) between \(x \text{ and } y\) of strength \(2\) if:

- \(c_1 = x\)
- \(c_n = y\)
- \(d(c_i, c_{i+1}) > d(c_{i+1}, c_i)\) for all \(i = 1, \ldots, n-1\)
- \(d(c_i, c_{i+1}) \geq 2\) for all \(i = 1, \ldots, n-1\) and there exist \(a \neq c_i, c_{i+1}\) s.t. \(d(c_i, c_{i+1}) = 2\).

Def Let \(p(x, y)\) be the maximal value \(a\) such that there exists a path of length \(a\) from \(x\) to \(y\), and \(p(x, y) = 0\) if no such path exists.

Then the Schulze winner is the Condorcet winner if it exists. Otherwise a potential winner is a candidate \(a\) such that:

\[p(a, x) \geq p(x, a)\]

Tie-breaking is used between potential winners.
Voters | 3 | 2 | 2 | 2
---|---|---|---|---
A, B | a | d | d | c
C, D | b | a | b | b
E, F | c | b | c | d
G, H | d | c | a | a

Condorcet winner? → no

d(x, y)

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-5</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>b</td>
<td>4</td>
<td>-7</td>
<td>5</td>
</tr>
<tr>
<td>c</td>
<td>4</td>
<td>2</td>
<td>-5</td>
</tr>
<tr>
<td>d</td>
<td>6</td>
<td>4</td>
<td>-4</td>
</tr>
</tbody>
</table>

\[ p(x, y) \]

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-5</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>b</td>
<td>5</td>
<td>-7</td>
<td>5</td>
</tr>
<tr>
<td>c</td>
<td>5</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>d</td>
<td>5</td>
<td>5</td>
<td>-5</td>
</tr>
</tbody>
</table>