Extensive Games

Action & Strategies

**Def:** Strategies

Let \( \Gamma = < N, A, H, P, (u_i) > \) be a EGWP. The set of actions \((h,a) \in H\) are denoted by \(A(h)\). Thus, a strategy of player \(i\) is a function \(s_i\) that assigns to each non-terminal history \(h \in H \setminus \emptyset\) with \(P(h) = i\) an action \(a \in A(h)\). The set of strategies of player \(i\) is denoted \(S_i\).

**Remark:** Strategies require us to assign actions to histories even if they are never played.
Notation: Strategies are often described by writing down the actions going through the game tree in breadth-first-search order (i.e., level-by-level, left to right).

Example

Strategies for player 1:
\( AE, \overline{AE}, BE, BF \)

Strategies for player 2:
\[ C, D \]
\[ AE \equiv \{ \emptyset \to A, (A, C) \to E \} \]

Def (Outcome)

The outcome of a strategy profile \( s = (s_i)_{i \in X} \) is the history \( h^s = (a_k)_{k \in K} \) such that for all \( 0 \leq k \leq K \), \( K \subseteq \mathbb{N} \cup \{ \infty \} \), where
\[ s_p(a_1, \ldots, a_k) = a_{k+1} \]
The outcome of $s$ is denoted by $O(s)$.

**Example**

$$O(AF, D) = (A, D)$$

$$O(AF, C) = (A, C, F)$$

**Nash Equilibrium in Extensive Games**

**Def (NE)**

A Nash equilibrium of an extensive game with perfect information $Γ$ is a strategy profile $s^* = (s_i^*)_{i ∈ N}$ such that for each player $i ∈ N$:

$$u_i(O(s^*)) ≥ u_i(O(s_{-i}^*, s_i^*))$$

for all $s_i ∈ S_i$. 
Proposition

The NE of an E6 wPI $\Gamma$ are exactly the NE of the strategic game induced by $\Gamma$ (called its strategic form) which is defined by

$$G' = \langle N', \{A'_i\}_{i \in N}, \{u'_i\}_{i \in N} \rangle$$

with

$$A'_i = S_i$$

$$u'_i(a) = u_i(0(s_i)),$$
Remarks

1) Each ECPWI can be transformed into a strategic game, but this can lead exponential blowup of the game representation.

2) The other direction does not hold because we do not have simultaneous moves in extensive games (yet).
Choosing B as player 1 is only plausible if he fears that player 2 might actually play L. But if player 1 chooses A, player 2 would never play L! For this reason, L is called a non-credible threat.
Subgame - Perfect Equilibrium

Let $\Gamma = \langle N, A, H, P, (u_i) \rangle$ be an EGWPI.

Def (Subgame)

The subgame of $\Gamma$ rooted at history $h$ is the EGWPI $\Gamma(h) = \langle N, A, H|_h, P|_h, (u_i|_h) \rangle$, where:

- $H|_h := \{ h' : (h, h') \in H \}$
- $P|_h := P((h, h'))$
- $u_i|_h := u_i((h, h'))$ for all $(h, h') \in Z$
For each strategy $s_i$ in $\Gamma$, let $s_i|_h (h') = s_i((h, h'))$ be the induced strategy in $\Gamma(h)$. The outcome function of $\Gamma(h)$ is denoted by $O_h$.

**Def (SPE)**

A **subgame-perfect equilibrium** (SPE) of a ELP $\Gamma$ is a strategy profile $S^* = (S_i^*)_{i \in N}$ such that for each history $h \in H$:

$$S^*/h = (S_i^*/h)_{i \in N}$$

is a NE of $\Gamma(h)$. 

\[ S = (A, R) : \]
\[ S = (\{ \emptyset \mapsto A \}, \{ (A) \mapsto R^3 \}) \]
\[ S \mid_{(A)} = (\emptyset \mapsto A) \mid_{(A)} : \{ (A) \mapsto R^3 \} \mid_{(A)} = ( \emptyset ) : \{ \emptyset \mapsto R^3 \} \]
\[ H = \emptyset, (A), (B), (A,L), (A,R) \]
\[ H \mid_{(A)} = \{ \emptyset, (L), (R) \} \]

\[ \Rightarrow \quad S = (A, R) \]
\[ h = \emptyset \quad S \text{ is a NE} \quad \Rightarrow \quad \text{SPE} \]
\[ h = (A) \quad S \mid_{(A)} \text{ is a NE} \quad \Rightarrow \quad \text{SPE} \]

\[ \Rightarrow \quad S = (B, L) \]
\[ h = \emptyset \quad S \text{ is a NE} \quad \Rightarrow \quad \text{not SPE} \]
\[ h = (A) \quad S \mid_{(A)} \text{ is not a NE} \quad \Rightarrow \quad \text{not SPE} \]
Example (Shapley game)

Possible profiles:

- (2:0, yyy)
- (2:0, yyn)
- (2:0, yny)
- (2:0, nny)

SPE: (1:1, yyy)

NE: (2:0, yyy)

Not NE: (2:0, nyy)
Questions

- Does an SPE always exist?
- Under which conditions?
- How to compute it?
- What is the complexity?

We show

- It is easy to verify that a profile is an SPE. ⇒ "one deviation property" (for finite horizon games)
- For finite games, we can easily compute the SPE by backward induction (Kuhn's Theorem)
Notation: If \( \Gamma \) is a \( \text{E}_6 \text{-WPI} \) then \( L(\Gamma) \) denotes the length of the longest history in \( \Gamma \).

Lemma (One Deviation Property)

Let \( \Gamma = < N, A, H, P, (u_i) > \) be a finite-horizon \( \text{E}_6 \text{-WPI} \). Then a strategy profile \( s^* \) is an SPE of \( \Gamma \) if and only if for every player \( i \in N \) and every history \( h \) of the profile \( p(h) = i \), we have

\[
\forall s_i, s_i \neq s_i^* \in A_i, u_i(h, s_i^* | h) = u_i(h, s_i^* | h)
\]

for every strategy \( s_i \) of player \( i \) in the subgame \( \Gamma(h) \) that differs from \( s_i^* \) only in the action after the initial history of \( \Gamma(h) \).