Extensive Games

Action and Strategies

Definition (Strategies)
Let $\Gamma = (N, A, H, P, (u_i))$ be an EGWP. The set of actions $(h, a) \in H$ are denoted by $A(h)$. Thus, a strategy of player $i$ is a function $s_i$ that assigns to each non-terminal history $h \in H \setminus \emptyset$ with $P(h) = a$ an action $a \in A(h)$. The set of strategies of player $i$ is denoted $S_i$.

Remark: Strategies require us to assign actions to histories even if they are never played.

The outcome of $s$ is denoted by $O(s)$.

Example

$O(\text{AF}, D) = (A, D)$

$O(\text{AF}, C) = (A, C, F)$

Nash Equilibrium in Extensive Games

Definition (NE)
A Nash equilibrium of an extensive game with perfect information $\Gamma$ is a strategy profile $s^* = (s_i^*)_{i \in N}$ such that for each player $i \in N$:

$u_i(s^*) \geq u_i(O(s^*, s_i))$ for all $s_i \in S_i$.

Notation: Strategies are often described by working down the action going through the game tree in breadth-first-search order (i.e., level-by-level). Left to right.

Example

Strategies for player $1$:
AE, AE, BE, BE

Strategies for player $2$:
C, D

AE $2 \{ \emptyset \rightarrow A, (A, C) \rightarrow E \}$

Definition (Outcome)
The outcome of a strategy profile $s = (s_i)_{i \in N}$ is the history $h^* = (a_{i_1}, a_{i_2})$ such that for all $D \in \text{K}$ with $s(D) = (a_{i_1}, a_{i_2}, \ldots, a_{i_N}) = D \in \text{K}$.

Proposition

The NE of an EGWP $\Gamma$ are exactly the NE of the strategic game induced by $\Gamma$ (called its strategic form) which is defined by

$G' = (N', (A'_i)_{i \in N'}, (u'_i)_{i \in N'})$ with

$A'_i = S_i$

$u'_i(a) = u_i(O(s_i^*))$. 
Remarks

1) Each EGwPI can be transformed into a strategic game, but this can lead to exponential blowup of the game representation.

2) The other direction does not hold, because we do not have simultaneous moves in extensive games (yet).

Subgame - perfect Equilibrium

Let $\Gamma = < N, A, H, I, P, U, >$ be an EGwPI.

Def (Subgame)

The subgame of $\Gamma$ rooted at history $h$ is the EGwPI $\Gamma(h) = < N, A_h, H|_h, P|_h, U|_h, >$, where:

- $H|_h = \{ h' : (h, h') \in H \}$
- $P|_h = P(\{ (h, h') \})$
- $U|_h = U(\{ (h, h') \})$ for all $(h, h') \in H$.

For each strategy $s_i$ in $\Gamma$, let $s_i|_h (h') = s_i((h, h'))$ be the induced strategy in $\Gamma(h)$.

The outcome function of $\Gamma(h)$ is denoted by $D_h$.

Def (SPE)

A subgame-perfect equilibrium (SPE) of an EGwPI $\Gamma$ is a strategy profile $s^* = (s^*_i)^i_{i=1}$ such that for each history $h \in H$:

- $s^*_i|_h : (s^*_i)_{i=1}^n$ is a NE of $\Gamma(h)$.
Questions:
- Does an SPE always exist?
- Under which conditions?
- How do we compute it?
- What is the complexity?

We show:
- It is easy to verify that a profile is an SPE, hence the deviation property (for finite-horizon games).
- For finite games, we can easily compute the SPE by backward induction (Kuhn's Theorem).

Example (Shapley game):

\[
S = (A, R): \quad S = \{ \emptyset \rightarrow A \} \quad \{ (A) \rightarrow R \}
\]

\[
S|_{(A)} = \{ \emptyset \rightarrow A \} \quad \{ (A) \rightarrow R \}
\]

\[
H = \emptyset, (A, (B, (A, L), (R))
\]

\[
H|_{(A)} = \emptyset, (L), (R)
\]

\[
\Rightarrow S = (A, R) \quad h = \emptyset \quad s is a NE \quad \Rightarrow \text{SPE}
\]

\[
H = (A) \quad s|_{(A)} \text{ is a NE}
\]

\[
\Rightarrow S = (B, C) \quad h = \emptyset \quad s is a NE \quad \Rightarrow \text{not SPE}
\]

Modulation: If \( \Gamma \) is a finite-horizon E6-WPI, then \( L(\Gamma) \) denotes the length of the longest history in \( \Gamma \).

Lemma (One Deviation Property):

Let \( \Gamma = \langle N, A, H, P, (u_i) \rangle \) be a finite-horizon E6-WPI. Then a strategy profile \( s^* \) is an SPE of \( \Gamma \) if and only if for every player \( i \in N \) and every history \( h \), for which \( P(h) = i \), we have

\[
u_i(h) (O_h (s^{|h} s_i^{|h})) \geq u_i(h) (O_h (s_i^{|h} s_i^{|h}))
\]

for every strategy \( s_i \) of player \( i \) in the subgame \( \Gamma(h) \) that differs from \( s_i^{|h} \) only in the action after the initial history of \( \Gamma(h) \).