Theorem: \text{Nash} is PPAD complete. \phantom{\square}

Some further results: Given a finite two-player game \(G\), it is \text{NP}-hard to decide whether there exists a Nash equilibrium \((\alpha, \beta)\) in \(G\) that has one of the following properties:

(a) player 1 (or 2) receives a payoff \(\geq k\).

(b) \(U_1(\alpha, \beta) + U_2(\alpha, \beta) \geq k\).

(c) \((\alpha, \beta)\) is Pareto optimal, i.e. there is no strategy profile \((\alpha', \beta')\) such that \(U_i(\alpha', \beta') \geq U_i(\alpha, \beta)\) for both \(i \in \{1, 2\}\), and \(U_i(\alpha', \beta') > U_i(\alpha, \beta)\) for some \(i \in \{1, 2\}\).
(d) player 1 (or player 2) plays some given a with probability > 0.

Extensive Games

So far: only simultaneous, one-shot games

Q: How to model sequential structure of many games (chess, ...)?

A: Use extensive game (≠ game trees).

Then, players have several choice points where they can decide how to play. Strategies then map chosen points to applicable actions.
Definition: An extensive game with perfect information (EGWPI) is a tuple $\Gamma = (N, A, H, P, (u_i)_{i\in N})$ where:

- $N$ is a finite, nonempty set of players.
- $A$ is a finite, nonempty set of actions.
- $H$ is a set of (finite or infinite) sequences over $A$ (called histories) such that:
  - the empty sequence $\langle \rangle \in H$,
  - if $\langle a_k \rangle_{k=1}^K \in H$ for some $K \in \mathbb{N} \cup \{0\}$ and $L < K$ then $\langle a_k \rangle_{k=1}^L \in H$ ("prefix closure")
  - if $\langle a_k \rangle_{k=1}^\infty$ is an action sequence such that $\langle a_k \rangle_{k=1}^L \in H$ for all $L \in \mathbb{N}$, then $\langle a_k \rangle_{k=1}^\infty \in H$. 

$P$ is a probability distribution on $H$.
A history is called terminal if it is infinite or if it is not a prefix of any longer history. The set of terminal histories is denoted by $Z$.

- $P : H \setminus Z \to N$ is the player function assigning to each nonterminal history $h \in H \setminus Z$ a player $P(h)$ whose turn it is to move in $h$.

- For each player $i \in N$, $u_i : Z \to \mathbb{R}$ is his utility function.

Terminology:

- $\Gamma$ is finite if $H$ is finite.
- $\Gamma$ has finite horizon if $H$ contains no infinite history.
Example (sharing game):

Task: Two players have to share two identical objects.

- Player 1 proposes an allocation.
- Player 2 accepts or declines.

\[ \text{objects are allocated as proposed} \quad \text{or} \quad \text{none gets anything} \]

Game tree:

\[ A = P(\langle \rangle) \]

\[ Z = P(\langle \langle 0, 2 \rangle \rangle) \]

\[ u_{\lambda}(\langle 2, 0 \rangle, y) = 2 \]
Formally: \( \Gamma = \langle N, D, H, P, (v_i): i \in N \rangle \) then

\[ N = \{1, 2, 3\}, \quad D = \{ (2,0), (4,1), (2,2), y, u \} \]

\[ H = \{ \langle y \rangle, \langle (2,0) \rangle, \langle (4,1) \rangle, \langle (2,2) \rangle, \langle (2,0), y \rangle, \langle (2,0), u \rangle, \ldots \} \]

\[ P(\langle y \rangle) = 1, \quad P(h) = 2 \quad \text{for all } h \in H \backslash \{2, 0, \langle y \rangle\} \]

\[ u_\gamma (\langle (2,0), y \rangle) = 2, \quad \text{etc.} \]