**Theorem (Nash):** Every finite strategic game has a MSNE. $\square$

**How to compute such MSNE?**

**Example:** MSNE in BoS

| | | p.2: B | p.2: S |
|---|---|---|---|
| p.1 | B | 2, 1 | 0, 0 |
| | S | 0, 0 | 1, 2 |

pure NE: (B, B), (S, S)

MSNE: here, in each truly mixed NE, $\mathrm{supp}(\alpha_1) = \mathrm{supp}(\alpha_2) = \{B, S\}$

Suppose that $(\alpha_1, \alpha_2)$ is a MSNE with $0 < \alpha_1(B) < 1$, $0 < \alpha_2(B) < 1$.

$$\overset{\text{Support lemma}}{\Longrightarrow} \quad \mathcal{U}_1(B, \alpha_2) = \mathcal{U}_1(S, \alpha_2)$$

$$\| \qquad\qquad\qquad \|$$

$$2 \cdot \alpha_2(B) + 0 \cdot \alpha_2(S) = 0 \cdot \alpha_2(B) + 1 \cdot \underbrace{\alpha_2(S)}_{1 - \alpha_2(B)}$$

$$\Leftrightarrow \quad 2\,\alpha_2(B) = 1 - \alpha_2(B)$$

$$\Leftrightarrow \quad 3\,\alpha_2(B) = 1 \quad \Rightarrow \quad \alpha_2(B) = \frac{1}{3} \Rightarrow \alpha_2(S) = \frac{2}{3}$$

Similarly: $\alpha_1(B) = \frac{2}{3}$ and $\alpha_1(S) = \frac{1}{3}$

$$\mathcal{U}_1(\alpha_1, \alpha_2) = \frac{2}{3} \cdot \frac{1}{3} \cdot 2 + 0 + 0 + \frac{1}{3} \cdot \frac{2}{3} \cdot 1 = \frac{2}{9} \cdot 3 = \frac{2}{3}$$

Similarly, $\mathcal{U}_2(\alpha_1, \alpha_2) = \frac{2}{3}$

**Systematic computation of MSNE:**

- For ZSG: Solve a LP
- For general-sum games: Solve a LCP.

**MSNE computation in finite ZSG**

Nash's theorem $\Rightarrow$ there exists at least one MSNE

$\overset{\text{NE vs. MM}}{\Longrightarrow}$ MSNE is a pair of MMs.

$\overset{\text{NE vs MM}}{\Longrightarrow}$ every pair of MMs is a MSNE, all with the same payoff profit

$\Longrightarrow$ it is sufficient to search for pairs of MMs.

**Background: Linear Programming**

**Goal:** Solving a system of linear inequalities over $n$ real variables with maximizing/minimizing some linear objective function.

**Example:**

| | | Cutting | Assembly | Postproc | Profit/item |
|---|---|---|---|---|---|
| 3 x | Sort 1 | 25 | 60 | 68 | 30 € |
| 5 y | Sort 2 | 75 | 60 | 34 | 40 € |
| | Constraint (day) | $\leq 450$ | $\leq 480$ | $\leq 476$ | ↑ Max! |

**Goal:** Find numbers of pieces $x$ (of sort 1) and $y$ (of sort 2) produced per day such that resource constraints are met and the objective function is maximized.

**Formalization:**

$$x \geq 0, \quad y \geq 0 \qquad (1)$$

$$25 \cdot x + 75 \cdot y \leq 450 \qquad (2)$$

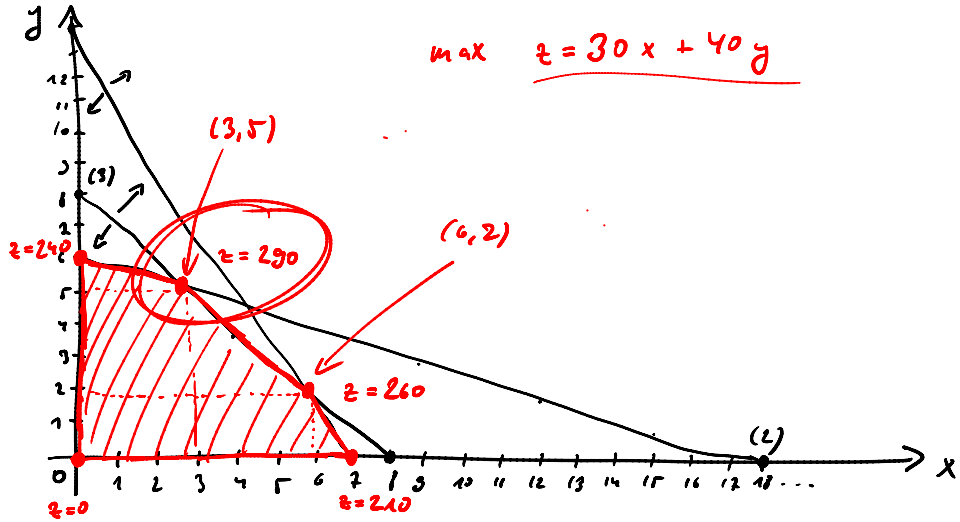
$$60 \cdot x + 60\, y \leq 480 \qquad (3)$$

$$68\, x + 34\, y \leq 476 \qquad (4)$$

maximize $30x + 40y$.

(1) - (4): feasible solutions

(5): objective function.




(2): $x + 3y \leq 18$ $\quad \boxed{y \leq 6 - \frac{1}{3}x}$

(3): $x + y \leq 8$ $\quad \boxed{y \leq 8 - x}$

(4): $2x + y \leq 14$, $y \leq 14 - 2x$

**Definition:** A linear program (LP) (in standard form) consists of

- $n$ real-valued variables $x_i$,
- $n$ coefficients $b_i$,
- $m$ constants $c_i$,
- $n \cdot m$ coefficients $a_{ji}$,
- $m$ inequalities $c_j \leq \sum_{i=1}^{n} a_{ji} x_i$, and
- an objective function $\sum_{i=1}^{n} b_i x_i$ $\quad (x_i \geq 0)$

to be minimized.

**Remarks:**

- Maximization instead of minimization: change the signs of all the $b_i$'s
- Equalities: $x + y \leq c$ iff ex. $z \geq 0$ s.t. $x + y + z = c$

  ($z$ is called a slack variable)

**LP solving algorithm(s):** Usually, one uses the simplex algorithm (worst-case exponential!); simplex algorithm typically preferred to existing polynomial-time algorithms such as interior point/ellipsoid algorithms. $\rightsquigarrow$ $\boxed{\text{lp\_solve}}$

## Encoding of finite ZSG MSNE as LP:

Let $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a finite ZSG, i.e.

- $N = \{1, 2\}$
- $A_1, A_2$ finite
- $u_1(\alpha) = -u_2(\alpha)$ for all $\alpha \in \Delta(A_1) \times \Delta(A_2)$.

Player 1 seeks a maximiniminizer mixed strategy $\alpha_1$:

For each $\alpha_1$ of player 1:

- determine utility under player 2's best response

Maximize over these utilities.

## LP constraints:

$$\boxed{\alpha_1(a_1)} \geq 0 \qquad \text{for all } a_1 \in A_1$$

$$\sum_{a_1 \in A_1} \boxed{\alpha_1(a_1)} = 1$$

$$\underbrace{\sum_{a_1 \in A_1} \boxed{\alpha_1(a_1)} \cdot u_1(a_1, a_2)}_{\mathcal{U}_1(\alpha_1, a_2)} \geq \boxed{u} \qquad \text{for all } a_2 \in A_2.$$

Maximize $u$.

- Solution to this LP is a MM for player 1.
- Solution to similar LP for pl. 2 is a MM for pl. 2.

## Example: Matching pennies

| | | p.2 H | p.2 T |
|---|---|---|---|
| p.1 | H | 1, −1 | −1, 1 |
| | T | −1, 1 | 1, −1 |

LP for player 1:

Maximize $u$ subject to these constraints:

$$\alpha_1(H) \geq 0, \quad \alpha_1(T) \geq 0,$$

$$\alpha_1(H) + \alpha_1(T) = 1$$

$$\alpha_1(H) - \alpha_1(T) \geq u$$

$$-\alpha_1(H) + \alpha_1(T) \geq u$$

Soln: $\alpha_1(H) = \alpha_1(T) = \frac{1}{2}, \quad u = 0$

**Remark:** Alternative (but similar) encoding using minimaximization instead maximiniminization possible.

LP with inequalities:

$$u_1(a_1, \alpha_2) \leq u \qquad \text{for all } a_1 \in A_1$$

minimize $u$. Similarly for $\alpha_1$.