**Theorem (Nash):** Every finite strategic game has a NE.

**How to compute such NE?**

**Example:** HSNE in Box

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>p:1</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>p:2</td>
<td></td>
<td>0, 0</td>
</tr>
</tbody>
</table>

Pure NE: (B, B), (S, S)

HSNE: here, in addition to pure NE, we require mixed NE, 

\[
\forall a, \frac{\text{supp}(a_i) \cap \text{supp}(a_j)}{\text{supp}(a_i) + \text{supp}(a_j)} = \frac{1}{2}, B, S
\]

Support that \((a_1, a_2)\) is a HSNE with 

\[0 < a_1(B) < 1, \quad 0 < a_2(S) < 1\]

\[
\text{Suppose } a_1(B) = \frac{1}{3}, \quad a_2(S) = \frac{2}{3}
\]

\[
U_1(a_1, a_2) = 0 \cdot a_1(B) + 0 \cdot a_2(S) = 0 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} = 0
\]

\[
U_2(a_1, a_2) = 2 \cdot a_1(B) + 0 \cdot a_2(S) = 2 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} = \frac{2}{3}
\]

Systematic computation of HSNE:

- For ZSG: Solve a LP
- For general-sum games: Solve a LCD

HSNE computation in finite ZSG

Nash's Theorem \(\implies\) there exists at least one HSNE

\[
\forall u, u_i \text{ is a pair of MTS} \implies \text{HSNE is a pair of MTSs.}
\]

\[
\forall u_i \text{ is a pair of MTSs, is a HSNE, all with the same payoff profile} \implies \text{it is sufficient to search for pairs of MTSs.}
\]

**Background: Linear Programming**

Goal: Solving a system of linear inequalities over a real variables with maximizing/minimizing some linear objective function.

**Example:**

<table>
<thead>
<tr>
<th></th>
<th>Cutting</th>
<th>Assembly</th>
<th>Postproc</th>
<th>Profit/item</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 x</td>
<td>25</td>
<td>60</td>
<td>68</td>
<td>30 $</td>
</tr>
<tr>
<td>5 y</td>
<td>75</td>
<td>60</td>
<td>34</td>
<td>40 $</td>
</tr>
</tbody>
</table>

| constraint (day) | \(\leq 450\) | \(\leq 480\) | \(\leq 426\) | Max |

\[\text{Max!}\]
Goal: Find number of pieces $x$ (of sort 1) and $y$ (of sort 2) produced per day so that resource constraints are met and the objective function is maximized.

Formulation:

\[ x \geq 0, \quad y \geq 0 \quad (1) \]
\[ 25x + 75y \leq 450 \quad (2) \]
\[ 60x + 60y \leq 480 \quad (3) \]
\[ 68x + 34y \leq 426 \quad (4) \]

maximize $30x + 140y$.

(1)-(4): feasible solutions

(5): objective function.

**Definition:** A linear program (LP) (a standard form) consists of

- $n$ real-valued variables $x_i$,
- $m$ coefficients $b_i$,
- $m$ constants $c_i$,
- $m$ coefficients $a_{ij}$,
- $m$ inequalities $c_j \leq \sum_{i=1}^{n} a_{ij} x_i$; and
- an objective function $\sum_{i=1}^{m} b_i x_i \quad (x \geq 0)$

\[ \text{to be minimized.} \]

**Remark:** Maximize instead of minimize:

- change the signs of all the $b_i$'s
- Equations: $x + y \leq c$ if $x, y \geq 0$

\[ \text{s.t.} \quad x + y + z = c \]

(z is called a slack variable)

**LP solving algorithm(s):** Usually, one uses the Simplex algorithm (which is explained!); another algorithm typically preferred to existing polynomial-time algorithms and is interior point elliptical algorithms. More → lp_solve
Encoding of finite 2SG PBE as LP:

Let $G = (N, (A_i))_{i=1}^n, (u_i)_{i=1}^n$ be a finite 2SG, i.e.

1. $N = \{1, 2\}$
2. $A_1, A_2$ finite
3. $u_a(x) = -u_a(x)$ for all $a \in A(A_1) \times A(A_2)$.

Player 1 seeks a maximum mixed strategy $\alpha_1$:

For each $a_2$ of player 2:

- determine utility under player 2's best response
- maximize over these utilities.

LP constraints:

- $\alpha_1(a_1) \geq 0$ for all $a_1 \in A_1$
- $\sum_{a_1 \in A_1} \alpha_1(a_1) = 1$
- $\sum_{a_2 \in A_2} \alpha_1(a_1) \cdot u_1(a_1, a_2) \geq u_1$ for all $a_2 \in A_2$.

Maximize $U$.

- Solution to this LP is a HH for player 1.
- Solution to similar LP for pl. 2 is a HH for pl. 2.

Example: Matching pennies

```
<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>-1,1</td>
<td>1,1</td>
</tr>
<tr>
<td>T</td>
<td>1,1</td>
<td>-1,1</td>
</tr>
</tbody>
</table>
```

LP for player 1:

- Maximize $u$ subject to these constraints:
  - $\alpha_1(H) \geq 0$, $\alpha_1(T) \geq 0$
  - $\alpha_1(H) + \alpha_1(T) = 1$
  - $\alpha_1(H) - \alpha_1(T) \geq u$
  - $-\alpha_1(H) + \alpha_1(T) \geq u$

Remarks: Alternative (but similar) encoding using minimization instead maximization possible.

LP with inequalities:

- $u_1(a_1, a_2) \leq u$ for all $a_2 \in A_2$
- Minimize $u$. Similarly for $\alpha_2$. 