Lem.: Let $G$ be a 2SG. Then
\[
\max_{y \in A_2} \min_{x \in A_1} u_2(x, y) = -\min_{y \in A_2} \max_{x \in A_1} u_1(x, y). \quad \square
\]

Maximin Minimax Theorem:
(a) $(x^*, y^*)$ is a NE of a 2SG $G$, $x^*$ and $y^*$ are MM of players 1 and 2, respectively.
(b) If $(x^*, y^*)$ is a NE of a 2SG $G$, then
\[
\max_{y \in A_2} \min_{x \in A_1} u_1(x, y) = \min_{x \in A_1} \max_{y \in A_2} u_2(x, y) = u_*(x^*, y^*).
\]
This means all NE of a 2SG have same payoffs.

Proof (PMM):
(c) Let $(x^*, y^*)$ be NE.
(i) NE $\Rightarrow$ $u_2(x^*, y^*) \geq u_2(x^*, y)$ f.o. $y \in A_2$
\[
\Rightarrow \quad u_1(x^*, y^*) \leq u_1(x^*, y) \text{ f.o. } y \in A_2
\]
\[
\Rightarrow \quad u_1(x^*, y^*) = \min_{y \in A_2} u_1(x^*, y)
\]
\[
= \max_{x \in A_1} \min_{y \in A_2} u_1(x, y) \quad \square
\]
(ii) NE $\Rightarrow$ $u_1(x^*, y^*) \geq u_1(x, y^*)$ f.o. $x \in A_1$
\[
\Rightarrow \quad u_2(x^*, y^*) \geq \min_{y \in A_2} u_2(x, y) \text{ f.o. } x \in A_1
\]
\[
\Rightarrow \quad u_2(x^*, y^*) = \max_{x \in A_1} \min_{y \in A_2} u_2(x, y)
\]
Corollary: If $(x_1^*, y_1^*)$ and $(x_2^*, y_2^*)$ are NE of 2SG $G$, then $(x_1^*, y_1^*)$ and $(x_2^*, y_2^*)$ are NE of $G$, too.

\[\Rightarrow \quad u_1(x_1^*, y_1^*) \geq \max_{x \in A_1} \min_{y \in A_2} u_1(x, y). \quad \square
\]
\[\Rightarrow \quad y_1^* \text{ is MM for player } 1.
\]

\[\Rightarrow \quad x_1^* \text{ is MM for player } 2.
\]
(b) From (c): \( u_1(x^*, y^*) = \max_{x \in \Theta_1} \min_{y \in \Theta_2} u_1(x, y) \)

and \( u_1(x^*, y^*) = -u_2(x^*, y^*) = -\max_{x \in \Theta_1} \min_{y \in \Theta_2} u_2(x, y) \)

\[ = \min_{y \in \Theta_2} \max_{x \in \Theta_1} u_1(x, y) \]

(c) Let \( x^* \) and \( y^* \) be NE for players 1/2,

and \( \max_{x \in \Theta_1} \min_{y \in \Theta_2} u_1(x, y) = \min_{y \in \Theta_2} \max_{x \in \Theta_1} u_1(x, y) =: \sigma^* \)

\[ \Rightarrow -\sigma^* = \max_{y \in \Theta_2} \min_{x \in \Theta_1} u_2(x, y) \]

Lemma \( \Rightarrow -\sigma^* = \max_{y \in \Theta_2} \min_{x \in \Theta_1} u_2(x, y) \).

\( x^*, y^* \) NE \( \Rightarrow \) \( u_1(x^*, y^*) \geq \sigma^* \) f.a. \( y \in \Theta_2 \)

and \( u_2(x^*, y^*) \geq -\sigma^* \) f.a. \( x \in \Theta_1 \).

\[ \Rightarrow u_1(x^*, y^*) \geq -\sigma^* \Rightarrow u_1(x^*, y^*) \leq \sigma^* \]

\[ \Rightarrow u_1(x^*, y^*) = \sigma^* \]

\[ \Rightarrow \sigma^* = u_1(x^*, y^*) \]

\( \Rightarrow \sigma^* \leq u_2(x^*, y^*) \)

\( \Rightarrow y^* \in \Theta_2(x^*) \).

\( \star \star \star \star \Rightarrow \) \( u_2(x^*, y^*) \geq u_2(x^*, y^*) \) f.a. \( y \in \Theta_2 \)

\[ \Rightarrow u_2(x^*, y^*) \leq u_2(x^*, y^*) \] f.a. \( y \in \Theta_2 \)

\[ \Rightarrow x^* \in \Theta_1(y^*) \]

\( \Rightarrow (x^*, y^*) \) NE.

Prove (Concl.) Let \( (x^*_1, y^*_n) \) and \( (x^*_2, y^*_2) \)

are NE.

1. With (a): \( x^*_1, x^*_2 \) NE for 1, \( y^*_n, y^*_2 \) NE f.a. 2.

2. With (b): \( \max_{x \in \Theta_1} \min_{y \in \Theta_2} u_1(x, y) = \max_{x \in \Theta_1} \min_{y \in \Theta_2} u_1(x, y) \)

3. With (a) and (b): \( (x^*_1, y^*_n) \) and \( (x^*_2, y^*_2) \) NE.