An object has to be assigned to one player i ∈ {1, ..., n} in exchange for a payment. For each player i, \( v_i \) is the valuation of player i of the object. W.l.o.g. we will assume that

\[ v_1 > v_2 > v_3 > ... > v_n. \]

**Mechanisms:** Player give simultaneously their bids \( b_1, b_2, ..., b_n \geq 0 \). The object is assigned to player with the highest bid. Break ties by valuation order, i.e., if \( b_i = b_j \) are the highest bids, then i wins if \( i < j \).
**First price auction:** The payment by the winner is his bid.

**Second price auction:** The payment by the winner is the highest bid of the non-winning agents.

Formalization of these games:

\( N = \{1, \ldots, n\} \)

\( A_i = \{ b_i \mid b_i \in \mathbb{R}^+ \} \)

\( u_i (b) = \begin{cases} 0 & \text{if the player } i \text{ does not win} \\ v_i - b_i & \text{otherwise} \end{cases} \)

For first price auction.
For second price auction:

- $N, A_i$; for each $i$ is the same

$$v_i(s) = \begin{cases} 0, & \text{if } i \text{ does not win} \\ v_i - \max b_{-i}, & \text{otherwise} \end{cases}$$

Example: Three players: 1, 2, 3.

- $v_1 = 100, v_2 = 80, v_3 = 53$
- $b_1 = 90, b_2 = 85, v_3 = 45$

1 wins and gets the utility:

- $v_1(b) = v_1 - b_1 = 10, 1$ if first price auction
- $u_1(b) = v_1 - \max b_{-1} = 15$ if second price auction
Proposition: In a second price auction, bidding your own valuation, \( b^+ \), is a weakly dominating strategy.

Proof:
1) Regardless of what the others agent do, \( b^+ \) is always the best strategy:

- If \( i \) wins: \( i \) has to pay \( \max b - v_i \), which means that \( v_i (b_i, b^+) \geq 0 \). Lowering our bid cannot improve the pay off, but might do lowering the auction. Increasing our bid does not help either.

- If \( i \) loses: \( v_i(b_i, b^+) = 0 \). Lowering \( b^+ \) does not change anything. By increasing his bid, he can win, he will have to pay a price \( \geq b^+ = v_i \), if either 0 or negative utility.
2) $b_i^+$ is strictly better than any other strategy under some profile.

Let $b_i$ some strategy $\neq b_i^+$.

$b_i < b_i^+$: Now let us consider $b_i$ with $\max b_i > b_i$. With $b_i$ we do not win, i.e., we have $v_i(b_i, b_i) = 0$.

While with $b_i^+$: $v_i(b_i, b_i^+) > 0$.

$b_i > b_i^+$: Consider $b_i > \max b_i > b_i^+$. Here

$v_i(b_i, b_i^+) < 0$ and $v_i(b_i, b_i^+) = 0$.

Remark: A profile of weakly dominated strategies is a NE, because for nobody there is an incentive to deviate.
Remark: There is a second NE for second price auctions! This is \( b = (v_1, v_1, ..., v_1) \), for \( -v_1 \): If he lowers, he does not win, so utility is still 0. If he increases, he still wins and has to pay \( v_1 \), utility is still 0.

For all others: Increasing leads to negative utility, decreasing does not change anything, since they do not win.
2.6 Zero Sum Games and NE

A zero sum game (ZSG) is a 2-player strategic game

\[ G = \langle \{1, 2\}, \{A_i\}_{i \in \{1, 2\}}, \{u_i\}_{i \in \{1, 2\}} \rangle \]

such that for all profiles \( a \in A \): \( u_1(a) + u_2(a) = 0 \).

Remark: Can be generalized to constant sum games, where the utilities sum up to some constant \( C \).
Idea: Try to play it safe.

Let us assume, the other player does as much harm as he can.

Then maximize over the outcome.

**Def** Let \( C \) be ZSG, \( x^* \in A_1 \) is called a maximinimizer for player 1, if:

\[
\min_{y \in A_2} v_1(x^*, y) \geq \min_{y \in A_2} v_2(x, y) \quad \text{for all } x \in A_1
\]

Similar for player 2.
Example

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1,7</td>
<td>2,2</td>
</tr>
<tr>
<td>B</td>
<td>-2,2</td>
<td>-4,4</td>
</tr>
</tbody>
</table>

Profile \( (T,L) \) is a NE, and it is a pair of maximizers. We will show that each NE in BSG is a pair of maximizers.

Lemma: Let \( G \) be a BSG. Then

\[
\max_{y \in A_2} \min_{x \in A_1} u_2(x,y) = -\min_{y \in A_2} \max_{x \in A_1} u_1(x,y)
\]

Proof: For each real valued function \( f \), it holds

\[
(1) \quad \min_{z} (-f(z)) = -\max_{z} (f(z))
\]
Thus:

\[
(2) \quad \min_{x \in A_2} u_2(x, y) = \max_{x \in A_2} -u_2(x, y)
\]

\[
\Rightarrow \quad u_1 = -u_2
\]

Thus:

\[
\max_{y \in A_2} \min_{x \in A_2} u_2(x, y) = -\min_{y \in A_2} -\left( \min_{x \in A_2} u_2(x, y) \right)
\]

\[
= -\min_{y \in A_2} \max_{x \in A_2} u_1(x, y)
\]
Maximinimizer Theorem

(a) Wherever \((x^*, y^*)\) is a NE of a ZSG, then \(x^*\) and \(y^*\) are maximinimizers of player 1 and player 2, respectively.

(b) If \((x^*, y^*)\) is a NE of a ZSG, then

\[
\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)
\]

\[
= u_1(x^*, y^*)
\]

This means all NE in ZSG have the same payoff.

(c) if \(\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)\) and \(x^*\) and \(y^*\) are maximinimizers for player 1 and player 2, respectively, then \((x^*, y^*)\) is a NE.
In particular, if \((x_1^*, y_1^*) \perp \cdots \perp (x_n^*, y_n^*)\) are NE, then so is \((x_1^*, y_1^*) \perp \cdots \perp (x_n^*, y_n^*)\).

Proof: