Sealed Bid Auctions

An object has to be assigned to one player i \in \{1, \ldots, n\} in exchange for a payment.
For each player i, \(v_i\) is the valuation of player i of the object. W.L.O.G. we will assume that
\[v_1 > v_2 > v_3 > \ldots > v_n.\]

Mechanisms: Player five simultaneously their bids \(b_1, b_2, \ldots, b_n \geq 0\). The object is assigned to player with the highest bid. Break ties by valuation order i.e. if \(b_i = b_j\) are the highest bids, then i will win if \(i < j\).

For second price auction:
\[N, A_i \text{ for each } i \text{ is the same} \]
\[u_i (b) = \begin{cases} 0 & \text{if } i \text{ does not win} \\ v_i - \max b_i & \text{other wise} \end{cases} \]

Example: Three players: 1, 2, 3.
\[v_1 = 100, v_2 = 80, v_3 = 55\]
\[b_1 = 90, b_2 = 85, b_3 = 45\]

A wins and gets the utility:
\[u_A (b) = v_A - b_A = 10\] for first price auction
\[u_A (b) = v_A - \max b_{-A} = 15\] for second price auction

First price auction: The payment by the winner is his bid.

Second price auction: The payment by the winner is the highest bid of the non-winning agents.

For illustration of these games:
\[N = \{1, \ldots, n\}\]
\[A_i = \{b_i \mid b_i \in \mathbb{R}^+\}\]
\[u_i (b) = \begin{cases} 0 & \text{if } \text{player } i \text{ does not win} \\ v_i - b_i & \text{other wise} \end{cases} \]

For first price auction:

Proposition: In a second price auction, bidding your own valuation, \(b_i^+\), is a weakly dominating strategy.

Proof:
1) Regardless of what the others agents do, \(b_i^+\) is always the best strategy.
   \[\text{i wins: } b_i^+ \text{ is always the lowest bid and means i would win }\]
   \[\text{i loses: } u_i (b_i^+, b_{-i}^+) = 0. \text{ Lowering } b_i^+ \text{ does not change the utility. By increasing bid bid, he can win, he will have to pay a price } \geq b_i^+ \text{ when he loses} \]

2) Other agents have to be careful.
2) \( b_i^+ \) is strictly better than any other strategy under some profile.
\[ b_i^+ \text{ some strategy} + b_i^+ \]

\( b_i^+ \): Now let us consider \( b_i^+ \) with \( \max b_i > b_i^+ \).
with \( b_i \) which does not win, i.e., we have \( v_i(b_i, b_i) = 0 \),
while with \( b_i^+ \): \( v_i(b_i^+, b_i) > 0 \).

\( b_i^- > b_i^+ \): Consider \( b_i^- > \max b_i > b_i^+ \). Here
\[ v_i(b_i^-, b_i^-) < 0 \text{ and } v_i(b_i^-, b_i^+) = 0. \]

Remark: A profile of weakly dominated strategies is a NE, because for nobody there is an incentive to deviate.

2.6 Zero Sum Games and NE

**Def (Zero sum game)**

A zero sum game \((2 \times 2)\) is a 2 player strategic game

\[ G = \langle \{A_1, A_2\}, \{b_1, b_2\}, \{u_1, u_2\} \rangle \]

such that for all profiles \( \pi \in \{A_1, A_2\}, \{b_1, b_2\} \)
\[ v_1(\pi) = -v_2(\pi) = 0. \]

Remark: Can be generalized to construct sum games, where the utilities sum up to some constant \( C \).

Remark: There is a second NE for second price auctions! This is \( b = (v_i, v_i, \ldots, v_i) \). For
\( v_i \): If he lowers, he does not win, so utility is still \( G \). If he increases, he still wins and has to pay \( v_i \), so utility is still \( G \).

For all others: Increasing leads to negative utility, decreasing does not change anything, since they do not win.

**Def**: Let \( G \) be \( 2 \times 2 \). \( x^* \in A_1 \) is called a maximum/min
for player 1 if:
\[ \min_{y \in A_2} v_1(x, y) \geq \min_{y \in A_2} v_1(x, y) \text{ for all } x \in A_1 \]

Similar for player 2.
Example

\[
\begin{array}{c|c|c|c}
T & 3 & 0 \\
3 & 2 & 1 \\
B & 2 & 0 & 2 \\
\end{array}
\]

Profile \((T, B)\) is a NE and it is a pair of maximinizers.

Lemma
Let \(C\) be a 2SC. Then

\[
\max \min_{y \in A_2} \min_{x \in A_1} u_2(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_2(x, y)
\]

Proof: For each real valued function \(f\), it holds

\[
\min_{x \in X} \left( -f(x) \right) = - \max_{x \in X} f(x)
\]

Maximum: 2NE Theorem

(a) Whenever \((x^*, y^*)\) is a NE of a 2SC \(C\), then \(x^*\)

and \(y^*\) are maximinizers of player 1 and player 2

respectively.

(b) If \((x^*, y^*)\) is a NE of a 2SC \(C\), then

\[
\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)
\]

This means all NE in 2SC have the same payoff.

(c) If \(\max_{x \in A_1} \min_{y \in A_2} u_1(x, y) = \min_{y \in A_2} \max_{x \in A_1} u_1(x, y)\)

and \(x^*\) and \(y^*\) are maximinizers for player 1 and

player 2, respectively, then \((x^*, y^*)\) is a NE.