Game Theory

17. Mechanisms with Money

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Research Group Foundations of Artificial Intelligence
July 1, 2013
Course evaluation:

- Yay! It’s time for the course evaluation once again!

- See
  
  http://www.informatik.uni-freiburg.de/~welte/lehrevaluation/ss2013/webseite_links.html

- Please complete the evaluation by July 15th, so we can discuss it on July 18th.

- Separate evaluations for separate lecturers.
Motivation

- Preference relations $\prec$ contain no information about “by how much” one candidate is preferred.
- Idea: Use money to measure this.
- Use money also for transfers between players “for compensation”.

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Formalization:

- Set of alternatives $A$.
- Set of $n$ players $I$.
- Valuation functions $v_i : A \rightarrow \mathbb{R}$ such that $v_i(a)$ denotes the value player $i$ assigns to alternative $a$.
- Payment functions specifying amount $m_i \in \mathbb{R}$ that player $i$ receives (or pays).
- Utility of player $i$: $u_i(a) = v_i(a) + m_i$. 
Second Price Auctions
Second Price Auctions

Recall second price auctions (cf. gametheory03.pdf):

- There are $n$ players bidding for a single item.
- Player $i$'s private valuations of item: $w_i$.
- Desired outcome: Player with highest private valuation wins bid.
- Players should reveal their valuations truthfully.
- Winner $i$ pays price $p^*$ and has utility $w_i - p^*$.
- Non-winners pay nothing and have utility 0.
Second Price Auctions

Formally:

- $A = N$
- $v_i(a) = \begin{cases} w_i & \text{if } a = i \\ 0 & \text{else} \end{cases}$

What about payments? Say player $i$ wins:

- $p^* = 0$ (winner pays nothing): bad idea, players would manipulate and publicly declare values $w'_i \gg w_i$.
- $p^* = w_i$ (winner pays his valuation): bad idea, players would manipulate and publicly declare values $w'_i = w_i - \varepsilon$.
- better: $p^* = \max_{j \neq i} w_j$ (winner pays second highest bid).
Vickrey Auction

Definition (Vickrey Auction)

The winner of the Vickrey Auction (aka second price auction) is the player $i$ with the highest declared value $w_i$. He has to pay the second highest declared bid $p^* = \max_{j \neq i} w_j$.

Proposition (Vickrey)

Let $i$ be one of the players and $w_i$ his valuation for the item, $u_i$ his utility if he truthfully declares $w_i$ as his valuation of the item, and $u'_i$ his utility if he falsely declares $w'_i$ as his valuation of the item. Then $u_i \geq u'_i$.

Proof

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Proof

Incentive Compatible Mechanisms
**Incentive Compatible Mechanisms**

- **Idea:** Generalization of Vickrey auctions.
- **Preferences** modeled as functions \( v_i : A \rightarrow \mathbb{R} \).
- Let \( V_i \) be the **space of all such functions** for player \( i \).
- Unlike for social choice functions: Not only decide about **chosen alternative**, but also about **payments**.
Mechanisms

Definition (Mechanism)

A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) consists of

- a social choice function \( f : V_1 \times \cdots \times V_n \rightarrow A \) and
- for each player \( i \), a payment function \( p_i : V_1 \times \cdots \times V_n \rightarrow \mathbb{R} \).

Definition (Incentive Compatibility)

A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) is called incentive compatible if for each player \( i = 1, \ldots, n \), for all preferences \( v_1 \in V_1, \ldots, v_n \in V_n \) and for each preference \( v'_i \in V_i \),

\[
v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}).
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Mechanisms

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A mechanism $\langle f, p_1, \ldots, p_n \rangle$ is called incentive compatible if for each player $i = 1, \ldots, n$, for all preferences $v_1 \in V_1, \ldots, v_n \in V_n$ and for each preference $v'_i \in V_i$, $v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i})$. 

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VCG Mechanisms
VCG Mechanisms

- If $\langle f, p_1, \ldots, p_n \rangle$ is incentive compatible, truthfully declaring ones preference is dominant strategy.

- The **Vickrey-Clarke-Groves mechanism** is an incentive compatible mechanism that maximizes “social welfare”, i.e., the sum over all individual utilities $\sum_{i=1}^{n} v_i(a)$.

- **Idea:** Reflect other players’ utilities in payment functions, align all players’ incentives with goal of maximizing social welfare.
Definition (Vickrey-Clarke-Groves mechanism)

A mechanism \( \langle f, p_1, \ldots, p_n \rangle \) is called a Vickrey-Clarke-Groves mechanism (VCG mechanism) if

1. \( f(v_1, \ldots, v_n) \in \arg\max_{a \in A} \sum_{i=1}^{n} v_i(a) \) for all \( v_1, \ldots, v_n \) and
2. there are functions \( h_1, \ldots, h_n \) with \( h_i : V_{-i} \rightarrow \mathbb{R} \) such that
   \[
   p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n))
   \]
   for all \( i = 1, \ldots, n \) and \( v_1, \ldots, v_n \).

Note: \( h_i(v_{-i}) \) independent of player \( i \)'s declared preference \( \Rightarrow h_i(v_{-i}) = c \) constant from player \( i \)'s perspective.

Utility of player \( i \) = \( v_i(f(v_1, \ldots, v_n)) + \sum_{j \neq i} v_j(f(v_1, \ldots, v_n)) - c = \sum_{j=1}^{n} v_j(f(v_1, \ldots, v_n)) - c = \text{social welfare} - c. \)
VCG Mechanisms

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Utility of player $i = v_i(f(v_1, \ldots, v_n)) + \sum_{j \neq i} v_j(f(v_1, \ldots, v_n)) - c = \sum_{j=1}^{n} v_j(f(v_1, \ldots, v_n)) - c = \text{social welfare} - c.$
Theorem (Vickrey-Clarke-Groves)

Every VCG mechanism is incentive compatible.

Proof

Let $i$, $v_i$, $v_i'$ and $v_{-i}$ be given. Show: Declaring true preference $v_i$ dominates declaring false preference $v_i'$.

Let $a = f(v_i, v_{-i})$ and $a' = f(v_i', v_{-i})$.

Utility player $i = \begin{cases} v_i(a) + \sum_{j \neq i} v_j(a) - h_i(v_{-i}) & \text{if declaring } v_i \\ v_i(a') + \sum_{j \neq i} v_j(a') - h_i(v_{-i}) & \text{if declaring } v_i' \end{cases}$

Alternative $a = f(v_i, v_{-i})$ maximizes social welfare

$\Rightarrow v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a')$.

$\Rightarrow v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i})$. 

\[ \square \]
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VCG Mechanisms

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Let \( i, v \_i, v_i \) and \( v_i' \) be given. Show: Declaring true preference \( v_i \) dominates declaring false preference \( v_i' \).

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\[ v_i(a) + \sum_{j \neq i} v_j(a) \geq v_i(a') + \sum_{j \neq i} v_j(a') \]

\[ v_i(f(v_i, v \_i)) - p_i(v_i, v \_i) \geq v_i(f(v_i', v \_i)) - p_i(v_i', v \_i). \]
Clarke Pivot Rule

- **So far:** payment functions $p_i$ and functions $h_i$ unspecified.

- **One possibility:** $h_i(v_{-i}) = 0$ for all $h_i$ and $v_{-i}$.
  
  **Drawback:** Too much money distributed among players (more than necessary).

- **Further requirements:**
  
  - Players should pay at most as much as they value the outcome.
  - Players should only pay, never receive money.
Definition (individual rationality)

A mechanism is **individually rational** if all players always get a nonnegative utility, i.e., if for all $i = 1, \ldots, n$ and all $v_1, \ldots, v_n$,

$$v_i(f(v_1, \ldots, v_n)) - p_i(v_1, \ldots, v_n) \geq 0.$$ 

Definition (positive transfers)

A mechanism has **no positive transfers** if no player is ever paid money, i.e., for all preferences $v_1, \ldots, v_n$,

$$p_i(v_1, \ldots, v_n) \geq 0.$$
Definition (Clarke pivot function)

The **Clarke pivot function** is the function

\[ h_i(v_{-i}) = \max_{b \in A} \sum_{j \neq i} v_j(b). \]

This leads to payment functions

\[ p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \]

for \( a = f(v_1, \ldots, v_n) \).

- Player \( i \) pays the difference between what the other players could achieve without him and what they achieve with him.
- Each player internalizes the externalities he causes.
Clarke Pivot Function

Example

- **Players** $I = \{1, 2\}$, **alternatives** $A = \{a, b\}$.
- **Values**: $v_1(a) = 10$, $v_1(b) = 2$, $v_2(a) = 9$ and $v_2(b) = 15$.
- Without player 1: $b$ best, since $v_2(b) = 15 > 9 = v_2(a)$.
- With player 1: $a$ best, since $v_1(a) + v_2(a) = 10 + 9 = 19 > 17 = 2 + 15 = v_1(b) + v_2(b)$.
- With player 1, other players (i.e., player 2) lose $v_2(b) - v_2(a) = 6$ units of utility.

$\Rightarrow$ **Clarke pivot function** $h_1(v_2) = 15$

$\Rightarrow$ **payment function**

$$p_1(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq 1} v_j(b) - \sum_{j \neq 1} v_j(a) = 15 - 9 = 6.$$
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- **With player 1,** other players (i.e., player 2) lose
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$$p_1(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq 1} v_j(b) - \sum_{j \neq 1} v_j(a) = 15 - 9 = 6.$$
Lemma (Clarke pivot rule)

A VCG mechanism with Clarke pivot functions has no positive transfers. If $v_i(a) \geq 0$ for all $i = 1, \ldots, n$, $v_i \in V_i$ and $a \in A$, then the mechanism is also individually rational.

Proof

Let $a = f(v_1, \ldots, v_n)$ be the alternative maximizing $\sum_{j=1}^{n} v_j(a)$, and $b$ the alternative maximizing $\sum_{j \neq i} v_j(b)$.

Utility of player $i$: $u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b)$.

Payment function for $i$: $p_i(v_1, \ldots, v_n) = \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$.

Since $b$ maximizes $\sum_{j \neq i} v_j(b)$: $p_i(v_1, \ldots, v_n) \geq 0$ (no positive transfers).
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Since \( b \) maximizes \( \sum_{j \neq i} v_j(b) \): \( p_i(v_1, \ldots, v_n) \geq 0 \) (no positive transfers).
Proof (ctd.)

Individual rationality: Since $v_i(b) \geq 0$,

$$u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j=1}^{n} v_j(a) - \sum_{j=1}^{n} v_j(b).$$

Since $a$ maximizes $\sum_{j=1}^{n} v_j(a)$,

$$\sum_{j=1}^{n} v_j(a) \geq \sum_{j=1}^{n} v_j(b)$$

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$$u_i = v_i(a) + \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(b) \geq \sum_{j=1}^{n} v_j(a) - \sum_{j=1}^{n} v_j(b).$$

Since $a$ maximizes $\sum_{j=1}^{n} v_j(a)$, 

$$\sum_{j=1}^{n} v_j(a) \geq \sum_{j=1}^{n} v_j(b)$$

and hence $u_i \geq 0$.

Therefore, the mechanism is also individually rational.
Vickrey Auction as a VCG Mechanism

- $A = N$. Valuations: $w_i, v_a(a) = w_a, v_i(a) = 0 (i \neq a)$.
- $a$ maximizes social welfare $\sum_{i=1}^{n} v_i(a)$ iff $a$ maximizes $w_a$.
- Let $a = f(v_1, \ldots, v_n) = \text{argmax}_{j \in A} w_j$ be the highest bidder.
- Payments: $p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$.
- But $\max_{b \in A} \sum_{j \neq i} v_j(b) = \max_{b \in A \setminus \{i\}} w_b$.
- Winner pays value of second highest bid:
  $$p_a(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq a} v_j(b) - \sum_{j \neq a} v_j(a)$$
  $$= \max_{b \in A \setminus \{a\}} w_b - 0 = \max_{b \in A \setminus \{a\}} w_b.$$
- Non-winners pay nothing: For $i \neq a$,
  $$p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$
  $$= \max_{b \in A \setminus \{i\}} w_b - w_a = w_a - w_a = 0.$$
Vickrey Auction as a VCG Mechanism

- A = N. Valuations: \( w_i \cdot v_a(a) = w_a, \ v_i(a) = 0 \ (i \neq a) \).
- \( a \) maximizes social welfare \( \sum_{i=1}^{n} v_i(a) \) iff \( a \) maximizes \( w_a \).
- Let \( a = f(v_1, \ldots, v_n) = \arg\max_j w_j \) be the highest bidder.
- Payments: \( p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) \).
- But \( \max_{b \in A} \sum_{j \neq i} v_j(b) = \max_{b \in A \{i\}} w_b \).
- Winner pays value of second highest bid:

\[
p_a(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq a} v_j(b) - \sum_{j \neq a} v_j(a) = \max_{b \in A \{a\}} w_b - 0 = \max_{b \in A \{a\}} w_b.
\]

- Non-winners pay nothing: For \( i \neq a \),

\[
p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a) = \max_{b \in A \{i\}} w_b - w_a = w_a - w_a = 0.
\]
Vickrey Auction as a VCG Mechanism

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Vickrey Auction as a VCG Mechanism

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  \[
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Vickrey Auction as a VCG Mechanism

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  $$p_a(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq a} v_j(b) - \sum_{j \neq a} v_j(a)$$

  $$= \max_{b \in A \setminus \{a\}} w_b - 0 = \max_{b \in A \setminus \{a\}} w_b.$$

- Non-winners pay nothing: For $i \neq a$,

  $$p_i(v_1, \ldots, v_n) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j(a)$$

  $$= \max_{b \in A \setminus \{i\}} w_b - w_a = w_a - w_a = 0.$$
Example: Bilateral Trade

- **Seller** $s$ offers item he values with $0 \leq w_s \leq 1$.
- **Potential buyer** $b$ values item with $0 \leq w_b \leq 1$.
- **Alternatives** $A = \{ \text{trade}, \text{no-trade} \}$.
- **Valuations:**
  \[
  v_s(\text{no-trade}) = 0, \quad v_s(\text{trade}) = -w_s, \\
  v_b(\text{no-trade}) = 0, \quad v_b(\text{trade}) = w_b.
  \]
- **VCG mechanism** maximizes $v_s(a) + v_b(a)$.
- **We have**
  \[
  v_s(\text{trade}) + v_b(\text{trade}) = w_b - w_s, \\
  v_s(\text{no-trade}) + v_b(\text{no-trade}) = 0
  \]
  i.e., **trade** maximizes social welfare iff $w_b \geq w_s$. 
Example: Bilateral Trade (ctd.)

- **Requirement:** if *no-trade* is chosen, neither player pays anything:
  \[ p_s(v_s, v_b) = p_b(v_s, v_b) = 0. \]

- To that end, choose Clarke pivot function for *buyer*:
  \[ h_b(v_s) = \max_{a \in A} v_s(a). \]

- **For seller:** Modify Clarke pivot function by an additive constant and set
  \[ h_s(v_b) = \max_{a \in A} v_b(a) - w_b. \]
Example: Bilateral Trade (ctd.)

For alternative **no-trade**,

\[
p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\text{no-trade})
\]

\[
= w_b - w_b - 0 = 0 \quad \text{and}
\]

\[
p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\text{no-trade})
\]

\[
= 0 - 0 = 0.
\]

For alternative **trade**,

\[
p_s(v_s, v_b) = \max_{a \in A} v_b(a) - w_b - v_b(\text{trade})
\]

\[
= w_b - w_b - w_b = -w_b \quad \text{and}
\]

\[
p_b(v_s, v_b) = \max_{a \in A} v_s(a) - v_s(\text{trade})
\]

\[
= 0 + w_s = w_s.
\]
Example: Bilateral Trade (ctd.)

- Because $w_b \geq w_s$, the seller gets at least as much as the buyer pays, i.e., the mechanism subsidizes the trade.
- Without subsidies, no incentive compatible bilateral trade possible.
- **Note**: Buyer and seller can exploit the system by colluding.
Example: Public Project

- Project costs $C$ units.
- Each citizen $i$ privately values the project at $w_i$ units.
- Government will undertake project if $\sum_i w_i > C$.
- Alternatives: $A = \{\text{project}, \text{no-project}\}$.
- Valuations:
  \[
  v_G(\text{project}) = -C, \quad v_G(\text{no-project}) = 0, \\
  v_i(\text{project}) = w_i, \quad v_i(\text{no-project}) = 0.
  \]
- VCG mechanism with Clarke pivot rule: for each citizen $i$,
  \[
  h_i(v_{-i}) = \max_{a \in A} \left( \sum_{j \neq i} v_j(a) + v_G(a) \right) \\
  = \begin{cases} 
  \sum_{j \neq i} w_j - C, & \text{if } \sum_{j \neq i} w_j > C \\
  0, & \text{otherwise}.
  \end{cases}
  \]
Example: Public Project (ctd.)

- Citizen $i$ pivotal if $\sum_j w_j > C$ and $\sum_{j \neq i} w_j \leq C$.
- Payment function for citizen $i$:

$$ p_i(v_1..n, v_G) = h_i(v_{-i}) - \left( \sum_{j \neq i} v_j \left( f(v_1..n, v_G) \right) + v_G \left( f(v_1..n, v_G) \right) \right) $$

- Case 1: Project undertaken, $i$ pivotal:

$$ p_i(v_1..n, v_G) = 0 - \left( \sum_{j \neq i} w_j - C \right) = C - \sum_{j \neq i} w_j $$

- Case 2: Project undertaken, $i$ not pivotal:

$$ p_i(v_1..n, v_G) = \left( \sum_{j \neq i} w_j - C \right) - \left( \sum_{j \neq i} w_j - C \right) = 0 $$

- Case 3: Project not undertaken:

$$ p_i(v_1..n, v_G) = 0 $$
Example: Public Project (ctd.)

- I.e., citizen \( i \) pays nonzero amount

\[
C - \sum_{j \neq i} w_j
\]

only if he is pivotal.

- He pays difference between value of project to fellow citizens and cost \( C \), in general less than \( w_i \).

- Generally,

\[
\sum_{i} p_i(\text{project}) \leq C
\]

i.e., project has to be subsidized.
Example: Buying a Path in a Network

- Communication network modeled as $G = (V, E)$.
- Each link $e \in E$ owned by different player $e$.
- Each link $e \in E$ has cost $c_e$ if used.
- **Objective**: procure communication path from $s$ to $t$.
- **Alternatives**: $A = \{ p \mid p \text{ path from } s \text{ to } t \}$.
- **Valuations**: $v_e(p) = -c_e$, if $e \in p$, and $v_e(p) = 0$, if $e \notin p$.
- **Maximizing social welfare**:
  - minimize $\sum_{e \in p} c_e$ over all paths $p$ from $s$ to $t$.
- **Example**:

  ![Graph](attachment:graph.png)

  - $c_a = 4$
  - $c_d = 12$
  - $c_b = 3$
  - $c_e = 5$
Example: Buying a Path in a Network (ctd.)

- For $G = (V, E)$ and $e \in E$ let $G \setminus e = (V, E \setminus \{e\})$.
- VGC mechanism with Clarke pivot function:
  \[
  h_e(v-e) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_{e'}
  \]
  i.e., the cost of the cheapest path from $s$ to $t$ in $G \setminus e$.
  (Assume that $G$ is 2-connected, s.t. such $p'$ exists.)
- Payment functions: for chosen path $p = f(v_1, \ldots, v_n)$,
  \[
  p_e(v_1, \ldots, v_n) = h_e(v-e) - \sum_{e \notin e' \in p} -c_{e'}.
  \]
  - **Case 1**: $e \notin p$. Then $p_e(v_1, \ldots, v_n) = 0$.
  - **Case 2**: $e \in p$. Then
    \[
    p_e(v_1, \ldots, v_n) = \max_{p' \in G \setminus e} \sum_{e' \in p'} -c_{e'} - \sum_{e \notin e' \in p} -c_{e'}.
    \]
Example: Buying a Path in a Network (ctd.)

Example:

\[ c_a = 4 \]
\[ c_b = 3 \]
\[ c_d = 12 \]
\[ c_e = 5 \]

- Cost along \( b \) and \( e \): 8
- Cost without \( e \): 3
- Cost of cheapest path without \( e \): 15 (along \( b \) and \( d \))
- Difference is payment: \(-15 - (-3) = -12\)
  - i.e., owner of arc \( e \) gets payed 12 for using his arc.

Note: Alternative path after deletion of \( e \) does not necessarily differ from original path at only one position. Could be totally different.
Summary

- New preference model: with money.
- VCG mechanisms generalize Vickrey auctions.
- VCG mechanisms are incentive compatible mechanisms maximizing social welfare.
- With Clarke pivot rule: even no positive transfers and individually rational (if nonnegative valuations).
- Various application areas.