Introduction to Game Theory
2. Strategic Games

Bernhard Nebel, Robert Mattmüller, Stefan Wölfl, & Christian Becker-Asano

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Boy's Contest:

Everybody chooses a number $n$ with $0 \leq n \leq 100$.

We consider the average $\phi$ of all chosen numbers.

Those players win who are next to $2/3 \cdot \phi$ (rounded up)!

<table>
<thead>
<tr>
<th>16</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>35</td>
<td>23</td>
</tr>
<tr>
<td>1</td>
<td>22</td>
<td>33</td>
</tr>
<tr>
<td>23</td>
<td>22</td>
<td>0</td>
</tr>
<tr>
<td>42</td>
<td>25</td>
<td>28</td>
</tr>
<tr>
<td>31</td>
<td>24</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>20</td>
<td>33</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>40</td>
</tr>
</tbody>
</table>

$\phi \approx 20, 16$
2.1 Definition

**Definition**  A strategic game is defined by a triple \( G = (N, (A_i)_{i \in N}, (\succ)_i) \)

with

(a) \( N \) is a finite set (set of agents/players)

(b) for each player \( i \in N \), a nonempty set \( A_i \) (the set of actions, strategies of player \( i \))

(c) for each player \( i \in N \), a preference relation \( \succ_i \) on the set \( A_i \) \( i.e., A_i \) is reflexive, transitive and complete.

\( A_i \) is the set of outcomes or action profiles or strategy profiles.
Sometimes the players' preferences are more naturally defined in terms of the consequences of their choices.

So assume we have a set of consequences $C$ and preference relations $(\succeq_i)$ in $\mathbb{N}$ defined on $C$. Let $g : A \rightarrow C$.

$$a \succeq_i b \Leftrightarrow g(a) \succeq_i g(b)$$

$(a, b \in A)$.

The set of consequences could, e.g., be a set of utility/expected profit profiles:

$$(u_1, \ldots, u_{|V|}) \in \mathbb{R}^{1\times |V|}$$
Alternative Def. of Strategic Games:

A strategic game is defined by

a triple \( G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle \)

with:

(a) and (b) as above

(c) for each player \( i \in N \) an utility function

\[ u_i : A := \prod_{i \in N} A_i \longrightarrow \mathbb{R} \]

Interpretation:

\[ a \succ_i b \iff u_i(a) \geq u_i(b) . \]
2.2 Payoff matrix

For 2 players and finite game $G$ we can represent strategic games as payoff matrices.

```
\begin{array}{c|cc}
\text{Players} & 1 & 2 \\
\hline
\text{T} & u_1, u_2 & u_1, u_2 \\
\text{B} & v_1, v_2 & 2_1, 2_2 \\
\end{array}
```

Profiles \((T, L), (T, R), (B, L), (B, R)\)

```
\begin{array}{c|c|c|c|c}
\text{Players} & 1 & 2 \\
\hline
\text{T} & u_1, u_2 & u_1, u_2 & u_1, u_2 & u_1, u_2 \\
\text{B} & v_1, v_2 & v_1, v_2 & v_1, v_2 & v_1, v_2 \\
\end{array}
```

\(u_1 = u_1(T, L), \quad \dot{z}_2 = u_2(B, R).\)
2.3 Examples

Player 1

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-2, -2</td>
<td>-2, -4.5</td>
<td>-1.5, -1.5</td>
</tr>
<tr>
<td>b</td>
<td>-1.5, -2</td>
<td>-2, -2</td>
<td>-1.5, -2</td>
</tr>
<tr>
<td>c</td>
<td>-1.5, -1.5</td>
<td>-2, -4.5</td>
<td>-2, -2</td>
</tr>
</tbody>
</table>

Player 2

s

\[
\begin{align*}
\frac{n_1}{n} & 1 \\
\frac{n_2}{n} & 0 \\
1 & \frac{n}{n}
\end{align*}
\]

\(s\) \rightarrow \text{it} \rightarrow \text{sink}
Prisoner's dilemma

Two suspects in a crime in separate cells.

* If both confess, each will be sentenced to 3 years in prison.
* If neither confesses, each will be sentenced to 4 years...
* If exactly one confesses, he will be freed, and the other sentenced to 4 years.

\[\begin{array}{c|cc}
& 
\text{Confess} & 
\text{Don't} \\
\hline
\text{Confess} & -3, -3 & 0, -4 \\
\text{Don't} & -4, 0 & -1, -1 \\
\end{array}\]
Nash Equilibrium:

Each of 2 persons chooses either 'Head' or 'Tail'. If the choices differ, person 1 pays 1 \$ to person 2. Otherwise, person 2 pays 1 \$ to person 1.

<table>
<thead>
<tr>
<th></th>
<th>Head</th>
<th>Tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>-1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>Tail</td>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

Example of a zero-sum game, i.e., a 2-player game in which for each profile \( a \in A \)

\[ u_1(a) + u_2(a) = 0 \]

Zero-sum games are a special case of strictly competitive games.

\[ a \preceq a \iff b \preceq b \preceq a. \]
2.4 Nash equilibrium

**Notations:**

Let 

\[ a = (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \in \prod_{i \in N} A_i = \Delta \]

be a strategy profile. We set

\[ a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) = (a_j)_{j \in N \setminus \{i\}} \]

\[ A_{-i} := \prod_{j \in N \setminus \{i\}} A_j \]

If \( a_{-i} \) is a strategy profile and \( a_i \in A_i \), then

\[ (a_{-i}, a_i) := (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \]

**Example:**

\[ a = (x, y, z) \implies a_{-2} = (x, z) \]

\[ a_2' = u \in A_2 \implies (a_{-2}, a_2') = (x, u, z) \]
Definition (Nash Equilibrium)

A Nash equilibrium (NE) of a strategic game $G$ is a strategy profile $a^* \in A^*$ such that for each $i \in V$,
\[ a^*_i = \left( a^*_i, a^*_i \right) \geq \left( a^*_i, a_i' \right) \text{ for all } a_i' \in A_i, \]

i.e. $a^*$ is a profile in which no player profits when he/she deviates from the profile.

Altehrerly,
\[ B^o_i \left( a_i \right) = \left\{ a_i' \in A_i : \left( a_i, a_i' \right) \geq \left( a_i, a_i \right) \forall a_i' \in A_i \right\} \]

called best response function. A NE, then, is any profile $a^*$ such that
\[ a^*_i \in B_i \left( a^*_i \right) \text{ for all } i \in V. \]

i.e. $a^* \in \prod_{i \in V} B_i \left( a^*_i \right)$. 
Prisoner's dilemma

<table>
<thead>
<tr>
<th>Confess</th>
<th>Don't</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 1</td>
<td>4, 0</td>
</tr>
<tr>
<td>0, 4</td>
<td>3, 3</td>
</tr>
</tbody>
</table>

Matching Pennies

<table>
<thead>
<tr>
<th>Head</th>
<th>Tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>-1, 1</td>
<td>1, -1</td>
</tr>
</tbody>
</table>

No NE!
2.5 Dominated Strategies

**Definition:**

A strategy $a_i^*$ of player $i$ is strictly dominated by strategy $a_i^{+}$ if for each profile $a_{-i}$ of the other agents $a_{-i} \in A_{-i}$,

$$(a_{-i}, a_i^{+}) > (a_{-i}, a_i^*)$$

<table>
<thead>
<tr>
<th></th>
<th>Conf.</th>
<th>Don’t</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Conf.</td>
<td>1, 1</td>
<td>4, 0</td>
<td>Head</td>
<td>1, 1</td>
<td>4, 0</td>
</tr>
<tr>
<td>Don’t</td>
<td>0, 4</td>
<td>3, 3</td>
<td>Tail</td>
<td>0, 4</td>
<td>3, 3</td>
</tr>
</tbody>
</table>
Definition:
A strategy \( a_i \in A_i \) is called _strictly dominated_ by \( a^+_i \in A_i \) if for each \( a_{-i} \in A_{-i} \),

\[(a_{-i}, a_i) \preceq_i (a_{-i}, a^+_i)\]

and if for at least one profile \( a_{-i} \in A_{-i} \):

\[(a_{-i}, a'_i) \prec_i (a_{-i}, a^+_i).\]

<table>
<thead>
<tr>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>2, 1</td>
</tr>
<tr>
<td>M</td>
<td>2, 1</td>
</tr>
<tr>
<td>B</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
References

Holler, Manfred J. & Gerhard Illing,  
*Einführung in die Spieltheorie*,  

Nisan, Noam, Tim Roughgarden, Éva Tardos, & Vijay V. Vazirani (eds.),  
*Algorithmic Game Theory*,  

Osborne, Martin J. & Ariel Rubinstein,  
*A Course in Game Theory*,  