Introduction to Game Theory

2. Strategic Games

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Beauty Contest:

Everybody chooses a number n with $0 \leq n \leq 100$.

We consider the average ϕ of all chosen numbers.

Those players win who are next to $2/3 \cdot \phi$ (rounded up)!

16	19	20
45	35	13
1	22	33
23	22	0
42	25	28
31	24	2
0	20	33
0	10	40

ϕ 20,16

<u>14</u>

2.1 Definition

<u>Definition</u>: A strategic game is defined by a triple $G = \langle N, (A_i)_{i \in N}, (\succsim)_{i \in N} \rangle$ with

(a) N is a finite set (set of agents/players)

Finite Game: all sets A_i are finite.

(b) for each player $i \in N$, a nonempty set $\underline{\underline{A_i}}$ (the set of actions, strategies of player i)

(c) for each player $i \in N$, a preference relation $\succsim_i$ on the set $A := \prod_{i \in N} A_i$, i.e. $\succsim_i$ is reflexive, transitive and complete.

$a \succ_i b \iff a \succsim_i b$ and $b \not\succsim_i a$

$a \sim_i b \iff a \succsim_i b$ and $b \succsim_i a$

A: the set of <u>outcomes</u> or <u>action profiles</u> or <u>strategy profiles</u>

Sometimes the players' preferences are more naturally defined in terms of the consequences of their choices.

So assume we have a set of consequences C and preference relations $(\succsim_i')_{i \in N}$ defined on C. Let $g: A \longrightarrow C$.

$$a \succsim_i b \;:\Longleftrightarrow\; g(a) \succsim_i' g(b)$$

$(a, b \in A)$.

The set of consequences could, e.g., be a set of utility / payoff profiles:

$$(u_1, \ldots, u_{|N|}) \in \mathbb{R}^{|N|}$$

Alternative Def. of Strategic Games:

A <u>strategic game</u> is defined by

a triple $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$

with:

(a) + (b) as above

(c) for each player $i \in N$ an utility function

$$u_i : A := \prod_{i \in N} A_i \longrightarrow \mathbb{R}$$

<u>Interpretation</u>:

$$a \succsim_i b \iff u_i(a) \geq u_i(b).$$

2.2 Payoff matrix

For 2 players and finite game G we can represent strategic games as payoff matrices

Player 2

Player 1		L	R
	T	w_1, w_2	x_1, x_2
	B	y_1, y_2	z_1, z_2

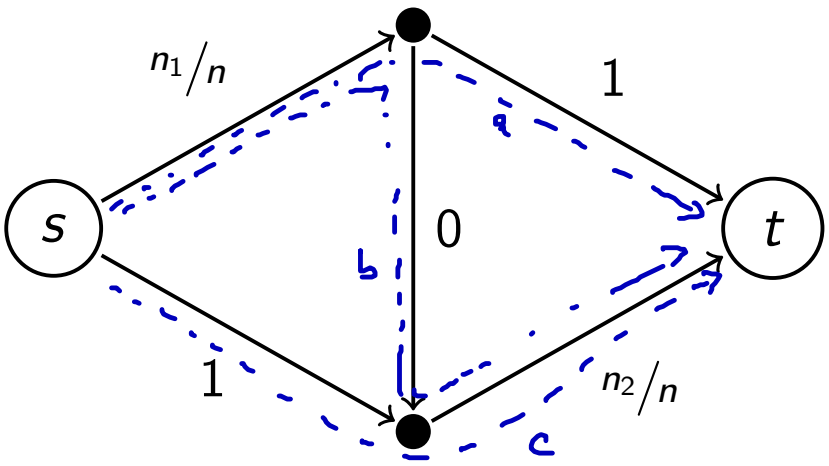
$w_1, w_2, x_1, x_2, \ldots \in \mathbb{R}$

<u>Profiles</u>

$$\begin{array}{cccc} (T,L) & (T,R), & (B,L) & (B,R) \\ \downarrow & \downarrow & \downarrow & \downarrow \\ (w_1,w_2) & (x_1,x_2) & (y_1,y_2) & (z_1,z_2) \end{array}$$

$w_1 = u_1(T,L)$. $z_2 = u_2(B,R)$.

2.3 Examples



Player 2

Player 1

	a	b	c
a	−2, −2	−2, −1.5	−1.5, −1.5
b	−1.5, −2	−2, −2	−1.5, −2
c	−1.5, −1.5	−2, −1.5	−2, −2

Prisoner's dilemma

Two suspects in a crime in separate cells.

* If both confess, each will be sentenced to 3 years in prison
* If neither confesses, each will be sentenced to 1 year ...
* If exactly one confesses, he will be freed, and the other sentenced to 4 years.

	Confess	Don't
Confess	-3, -3	0, -4
Don't	-4, 0	-1, -1

	Confess	Don't
Confess	1, 1	4, 0
Don't	0, 4	3, 3

Equivalent representations

Matching Pennies :

Each of 2 persons chooses either "Head" or "Tail". If the choices differ, person 1 pays 1 \$ to person 2. Otherwise, person 2 pays 1 \$ to person 1.

	Head	Tail
Head	1, −1	−1, 1
Tail	−1, 1	1, −1

Example of a zerosum game, i.e., a 2 player game in which for each profile $a \in A$

$$u_1(a) + u_2(a) = 0$$

Zerosum games are a special case of strictly competitive games.

$$a \precsim_1 b \iff b \precsim_2 a.$$

2.4 Nash equilibrium

<u>Notations:</u> Let $a = (a_1, \dots, a_{|N|}) \in \prod_{i \in N} A_i =: A$ be a strategy profile. We set

$$a_{-i} := (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{|N|}) = (a_j)_{j \in N \setminus \{i\}}$$

$$A_{-i} := \prod_{j \in N \setminus \{i\}} A_j$$

If a_{-i} is a strategy profile and $a_i \in A_i$, then

$$(a_{-i}, a_i) := (a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{|N|})$$

<u>Example:</u>

$$a = (X, Y, Z) \rightsquigarrow a_{-2} = (X, Z)$$

$$a_2' = U \in A_2 \rightsquigarrow (a_{-2}, a_2') = (X, U, Z)$$

Definition (Nash Equilibrium)

A Nash equilibrium (NE) of a strategic game G is a strategy profile $a^* \in A$ such that for each $i \in N$,

$$a^* = (a^*_{-i}, a^*_i) \succsim_i (a^*_{-i}, a'_i) \quad \text{for all } a'_i \in A_i,$$

i.e. a^* is a profile in which no player profits when he/she deviates from the profile.

Alternatively,

$$B_i(a_{-i}) := \{ a_i \in A_i : (a_{-i}, a_i) \succsim_i (a_{-i}, a'_i) \ \forall a' \in A_i \}$$

called best response function. A NE, then, is any profile a^* such that

$$a^*_i \in B_i(a^*_{-i}) \quad \text{for all } i \in N.$$

i.e. $a^* \in B(a^*) := \prod_{i \in N} B_i(a^*_{-i})$.

Prisoner's dilemma

NE unique NE

	Confess	Don't
Confess	$-1, 1$	$4, 0$
Don't	$0, 4$	$3, 3$

Matching Pennies

	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

No NE !

2.5 Dominated Strategies

Definition:

A strategy $a_i' \in A_i$ is strictly dominated by strategy $a_i^+ \in A_i$ if for each profile of the other agents $a_{-i} \in A_{-i}$,

$$(a_{-i}, a_i') \prec_i (a_{-i}, a_i^+)$$

NE

	Conf.	Don't
Conf.	1, 1	4, 0
~~Don't~~	~~0, 4~~	~~3, 3~~

	Head	Tail
Head	1, 1	4, 0
Tail	0, 4	3, 3

Definition:

A strategy $a_i^1 \in A_i$ is called <u>weakly</u> <u>dominated</u> by $a_i^+ \in A_i$ if for each $a_{-i} \in A_{-i}$,

$$(a_{-i}, a_i^1) \precsim_i (a_{-i}, a_i^+)$$

and if for at least one profile $a_{-i} \in A_{-i}$

$$(a_{-i}, a_i^1) \prec_i (a_{-i}, a_i^+).$$

	L	R
T	2 , 1	0 , 0
M	2 , 1	1 , 1
B	0 , 0	1 , 1

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