Exercise 0.1 - Proof by induction

1. Prove by induction that
   \[\sum_{i=0}^{n} i^2 = \frac{n \cdot (n + 1) \cdot (2n + 1)}{6}\]
   for any \(n \in \mathbb{N}\).

2. Consider the definition of a binary tree. Either, a tree is
   (a) a leaf, written \(\square\),
   (b) or an inner node with two children \(t_1\) and \(t_2\), which are both binary trees, written \(N(t_1,t_2)\).

   The number of inner nodes of a binary tree \(I(t)\) is defined as:
   \[I(t) = \begin{cases} 
   0 & \text{if } t = \square \\
   I(t_1) + I(t_2) + 1 & \text{if } t = N(t_1,t_2)
   \end{cases}\]

   The number of leaves of a tree \(t\) is defined by
   \[L(t) = \begin{cases} 
   1 & \text{if } t = \square \\
   L(t_1) + L(t_2) & \text{if } t = N(t_1,t_2)
   \end{cases}\]

   Prove that the difference between the number of leaves and the number of internal nodes in a binary tree is 1.

Exercise 0.2 - Complexity

Characterize the relationship between \(f(n)\) and \(g(n)\) in the following examples using the \(\mathcal{O}\)-, \(\Theta\)-
or \(\Omega\)-notation.

1. \(f(n) = n^{0.99998}\) \hspace{1cm} \(g(n) = \sqrt{n}\)
2. \(f(n) = 2\log^2(n)\) \hspace{1cm} \(g(n) = \sum_{k=1}^{n} \frac{n}{2^k}\)
3. \(f(n) = n \log_2(n)\) \hspace{1cm} \(g(n) = \sqrt{n}\)
4. \(f(n) = \sqrt{n}\) \hspace{1cm} \(g(n) = 1000n\)

Exercise 0.3 - Complexity

In order to solve a certain problem, five different algorithms \(A_1, \ldots, A_5\) were developed. Algorithm \(A_i\) needs \(T_i(n)\) time steps to solve the problem for an instance of size \(n\).

1. \(T_1(n) = 1000n\)
2. \(T_2(n) = 500n \log_2(n)\)
3. \(T_3(n) = n \sqrt{n}\)
4. \(T_4(n) = 10n^3\)
5. $T_5(n) = 2^n$

The algorithms will be executed on a Pentium 1GHz processor. For simplicity, we assume that the processor executes exactly $10^9$ computations per second. Compute for any $i$ the input size $n$, for which the problem can be solved by algorithm $A_i$ within 1h.

**Exercise 0.4 - Complexity classes**

Given the following classes, $\text{DLOG, PSPACE, PTIME, NP, coNP, NLOG}$ order them by set inclusions. (e.g. $\text{NLOG \subseteq PTIME}$)