Principles of Knowledge Representation and Reasoning

Nonmonotonic Reasoning III: Cumulative Logics

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Motivation

- Conventional NM logics are based on (ad hoc) modifications of the logical machinery (proofs/models).
- \textit{Nonmonotonicity} is only a negative characterization: If we have $\Theta \models \varphi$, we do not necessarily have $\Theta \cup \{\psi\} \models \varphi$.
- Could we have a constructive \textbf{positive} characterization of default reasoning?
Plausible Consequences

- In conventional logic, we have the logical consequence relation $\alpha \models \beta$: If $\alpha$ is true, then also $\beta$ is true.
- Instead, we will study the relation of plausible consequence $\alpha \triangleright \beta$: if $\alpha$ is all we know, can we conclude $\beta$?
- $\alpha \triangleright \beta$ does not imply $\alpha \land \alpha' \triangleright \beta$!
  Compare to conditional probability: $P(\beta|\alpha) \neq P(\beta|\alpha,\alpha')$!
- Find rules characterizing $\triangleright$: for example, if $\alpha \triangleright \beta$ and $\alpha \triangleright \gamma$, then $\alpha \triangleright \beta \land \gamma$.
- Write down all such rules!
- Perhaps we find a semantic characterization of $\triangleright$. 

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Instead, we will study the relation of plausible consequence $\alpha \triangleright \beta$: if $\alpha$ is all we know, can we conclude $\beta$?

$\alpha \triangleright \beta$ does not imply $\alpha \land \alpha' \triangleright \beta$!

Compare to conditional probability: $P(\beta|\alpha) \neq P(\beta|\alpha,\alpha')$!

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Write down all such rules!

Perhaps we find a semantic characterization of $\triangleright$. 

Desirable Properties 1: Reflexivity

- **Reflexivity:**
  
  $\alpha \sim \alpha$

  - **Rationale:** If $\alpha$ holds, this *normally implies* $\alpha$.
  - **Example:** Tom goes to a party *normally implies* that Tom goes to a party.
Reflexivity in Default Logic

Plausible consequence as Reasoning in Default Logic

Let us consider relations $\sim_\Delta$ that are defined in terms of Default Logic.

$\alpha \sim_{\langle D, W \rangle} \beta$ means that $\beta$ is a skeptical conclusion of $\langle D, W \cup \{\alpha\} \rangle$.

Proposition

Default Logic satisfies Reflexivity.

Proof.

The question is: does $\alpha$ skeptically follow from $\Delta = \langle D, W \cup \{\alpha\} \rangle$?

For all extensions $E$ of $\Delta$, $W \cup \{\alpha\} \subseteq E$ by definition. Hence $\alpha \in E$ and $\alpha$ belongs to all extensions of $\Delta$. 

\[\square\]
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Desirable Properties 2: Left Logical Equivalence

- **Left Logical Equivalence:**

  \[ \models \alpha \leftrightarrow \beta, \, \alpha \vdash \sim \gamma \]

  \[ \beta \vdash \sim \gamma \]

- **Rationale:** It is not the syntactic form, but the logical content that is responsible for what we conclude normally.

- **Example:** Assume that Tom goes or Peter goes normally implies Mary goes. Then we would expect that Peter goes or Tom goes normally implies Mary goes.
Proposition

*Default Logic satisfies Left Logical Equivalence.*

Proof.

Assume that $\models \alpha \leftrightarrow \beta$ and $\gamma$ is in all extensions of $\langle D, W \cup \{\alpha\} \rangle$. The definition of extensions is invariant under replacing any formula by an equivalent formula. Hence $\langle D, W \cup \{\beta\} \rangle$ has exactly the same extensions, and $\gamma$ is in every one of them.
Left Logical Equivalence in Default Logic

Proposition

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Desirable Properties 3: Right Weakening

- **Right Weakening:**

  \[ \vdash \alpha \rightarrow \beta, \gamma \not\beta, \alpha \not\beta \]

  \[ \gamma \not\beta \]

- **Rationale:** If something can be concluded normally, then everything classically implied should also be concluded normally.

- **Example:** Assume that

  Mary goes *normally implies* Clive goes and John goes.

  Then we would expect that

  Mary goes *normally implies* Clive goes.

- **From 1 & 3 supraclassicality follows:**

  \[ \alpha \not\beta \]

  \[ \frac{\vdash \alpha \rightarrow \beta, \alpha \not\beta}{\alpha \not\beta} \]

  \[ \frac{\alpha \not\beta}{\alpha \not\beta} \]
Right Weakening in Default Logic

Proposition

Default Logic satisfies Right Weakening.

Proof.

Assume $\alpha$ is in all extensions of a default theory $\langle D, W \cup \{\gamma\} \rangle$ and $\models \alpha \rightarrow \beta$. Extensions are closed under logical consequence. Hence also $\beta$ is in all extensions.
Proposition

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Desirable Properties 4: Cut

- **Cut:**

\[
\alpha \vdash \beta, \alpha \land \beta \vdash \gamma \\
\frac{\alpha \vdash \gamma}{\alpha \vdash \gamma}
\]

- **Rationale:** If part of the premise is plausibly implied by another part of the premise, then the latter is enough for the plausible conclusion.

- **Example:** Assume that John goes *normally implies* Mary goes. Assume further that John goes *and* Mary goes *normally implies* Clive goes. Then we would expect that John goes *normally implies* Clive goes.
Proposition

*Default Logic satisfies Cut.*

Proof idea.

Show that every extension $E$ of $\Delta = \langle D, W \cup \{\alpha}\rangle$ is also an extension of $\Delta' = \langle D, W \cup \{\alpha \land \beta}\rangle$.

Consistency of justifications of defaults is tested against $E$ both in the $W \cup \{\alpha\}$ case and in the $W \cup \{\alpha \land \beta\}$ case. The preconditions that are derivable when starting from $W \cup \{\alpha\}$ are also derivable when starting from $W \cup \{\alpha \land \beta\}$. $W \cup \{\alpha \land \beta\}$ does not allow deriving further preconditions because also with $W \cup \{\alpha\}$ at some point $\beta$ is derived. Hence $E$ is also an extension of $\Delta'$. Hence, because $\gamma$ belongs to all extensions of $\Delta'$, it also belongs to all extensions of $\Delta$. 
Cut in Default Logic

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Hence $E$ is also an extension of $\Delta'$.

Hence, because $\gamma$ belongs to all extensions of $\Delta'$, it also belongs to all extensions of $\Delta$. 
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Desirable Properties 5: Cautious Monotonicity

- **Cautious Monotonicity:**

\[
\frac{\alpha \not\vdash \beta, \alpha \not\vdash \gamma}{\alpha \land \beta \not\vdash \gamma}
\]

- **Rationale:** In general, adding new premises may cancel some conclusions. However, existing conclusions may be added to the premises without canceling any conclusions!

- **Example:** Assume that
  Mary goes *normally implies* Clive goes and
  Mary goes *normally implies* John goes.
  Mary goes *and* Jack goes might not *normally imply* that
  John goes.
  However, Mary goes and Clive goes should *normally imply* that John goes.
Proposition

Default Logic does not satisfy Cautious Monotonicity.

Proof.

Consider the default theory \( \langle D, W \rangle \) with

\[
D = \left\{ \frac{a: g}{g}, \frac{g: b}{b}, \frac{b: \neg g}{\neg g} \right\}
\]

and \( W = \{a\} \).

\( E = \text{Th}(\{a, b, g\}) \) is the only extension of \( \langle D, W \rangle \) and \( g \) follows skeptically.

For \( \langle D, W \cup \{b\} \rangle \) also \( \text{Th}(\{a, b, \neg g\}) \) is an extension, and \( g \) does not follow skeptically.
Lemma

*Rules 4 & 5 can be equivalently stated as follows.*

If $\alpha \not\vdash \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.

The above property is also called *cumulativity*.

**Proof.**

$\Rightarrow$: Assume that 4 & 5 hold and $\alpha \not\vdash \beta$. Assume further that $\alpha \not\vdash \gamma$. With rule 5 (CM), we have $\alpha \land \beta \not\vdash \gamma$. Similarly, from $\alpha \land \beta \not\vdash \gamma$ by rule 4 (Cut) we get $\alpha \not\vdash \gamma$.

Hence the plausible conclusions from $\alpha$ and $\alpha \land \beta$ are the same.

$\Leftarrow$. Assume *Cumulativity* and $\alpha \not\vdash \beta$. Now we can derive *rules 4 and 5*. 
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\[ \text{If } \alpha \vdash \beta, \text{ then the sets of plausible conclusions from } \alpha \text{ and } \alpha \land \beta \text{ are identical}. \]

The above property is also called **cumulativity**.

Proof.

\[ \Rightarrow: \text{ Assume that 4 & 5 hold and } \alpha \vdash \beta. \text{ Assume further that } \alpha \vdash \gamma. \] With **rule 5** (CM), we have \( \alpha \land \beta \vdash \gamma \). Similarly, from \( \alpha \land \beta \vdash \gamma \) by rules 4 and 5. Hence the sets of conclusions are the same.

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$\Leftarrow$: Assume Cumulativity and $\alpha \not\sim \beta$. Now we can derive rules 4 and 5.
Lemma

Rules 4 & 5 can be equivalently stated as follows.

If $\alpha \nvdash \beta$, then the sets of plausible conclusions from $\alpha$ and $\alpha \land \beta$ are identical.

The above property is also called cumulativity.

Proof.

$\Rightarrow$: Assume that 4 & 5 hold and $\alpha \nvdash \beta$. Assume further that $\alpha \nvdash \gamma$. With rule 5 (CM), we have $\alpha \land \beta \nvdash \gamma$. Similarly, from $\alpha \land \beta \nvdash \gamma$ by rule 4 (Cut) we get $\alpha \nvdash \gamma$. Hence the plausible conclusions from $\alpha$ and $\alpha \land \beta$ are the same.

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Hence the plausible conclusions from $\alpha$ and $\alpha \land \beta$ are the same.

$\Leftarrow$: Assume Cumulativity and $\alpha \not\sim \beta$. Now we can derive rules 4 and 5.
The System **C**

1. **Reflexivity**

   \[
   \alpha \sim \alpha
   \]

2. **Left Logical Equivalence**

   \[
   \vdash \alpha \leftrightarrow \beta, \alpha \sim \gamma \\
   \beta \sim \gamma
   \]

3. **Right Weakening**

   \[
   \vdash \alpha \rightarrow \beta, \gamma \sim \alpha \\
   \gamma \sim \beta
   \]

4. **Cut**

   \[
   \alpha \sim \beta, \alpha \land \beta \sim \gamma \\
   \alpha \sim \gamma
   \]

5. **Cautious Monotonicity**

   \[
   \alpha \sim \beta, \alpha \sim \gamma \\
   \alpha \land \beta \sim \gamma
   \]
Derived Rules in C

- **Equivalence:**
  \[
  \frac{\alpha \not\sim \beta, \beta \not\sim \alpha, \alpha \not\sim \gamma}{\beta \not\sim \gamma}
  \]

- **And:**
  \[
  \frac{\alpha \not\sim \beta, \alpha \not\sim \gamma}{\alpha \not\sim \beta \land \gamma}
  \]

- **MPC:**
  \[
  \frac{\alpha \not\sim \beta \rightarrow \gamma, \alpha \not\sim \beta}{\alpha \not\sim \gamma}
  \]
### Equivalence

| Assumption: | $\alpha \not\models \beta$, $\beta \not\models \alpha$, $\alpha \not\models \gamma$ |
| Cautious Monotonicity: | $\alpha \land \beta \not\models \gamma$ |
| Left L Equivalence: | $\beta \land \alpha \not\models \gamma$ |
| Cut: | $\beta \not\models \gamma$ |

### And

| Assumption: | $\alpha \not\models \beta$, $\alpha \not\models \gamma$ |
| Cautious Monotonicity: | $\alpha \land \beta \not\models \gamma$ |
| Propositional logic: | $\alpha \land \beta \land \gamma \models \beta \land \gamma$ |
| Supraclassicality: | $\alpha \land \beta \land \gamma \not\models \beta \land \gamma$ |
| Cut: | $\alpha \land \beta \not\models \beta \land \gamma$ |
| Cut: | $\alpha \not\models \beta \land \gamma$ |

**MPC** is an Exercise.
Proofs

<table>
<thead>
<tr>
<th>Equivalence</th>
</tr>
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<tbody>
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# Proofs

## Equivalence

| Assumption: | $\alpha \not\sim \beta$, $\beta \not\sim \alpha$, $\alpha \not\sim \gamma$ |
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| Cut: | $\beta \not\sim \gamma$ |

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**Assumption:** \( \alpha \not\sim \beta, \ \beta \not\sim \alpha, \ \alpha \not\sim \gamma \)

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**Left L Equivalence:** \( \beta \land \alpha \not\sim \gamma \)

**Cut:** \( \beta \not\sim \gamma \)

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**Propositional Logic:** \( \alpha \land \beta \land \gamma \models \beta \land \gamma \)

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**MPC is an Exercise.**
## Proofs

### Equivalence

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**Cautious Monotonicity:** \( \alpha \land \beta \sim \gamma \)

**Left L Equivalence:** \( \beta \land \alpha \sim \gamma \)

**Cut:** \( \beta \sim \gamma \)

### And

**Assumption:** \( \alpha \sim \beta, \ \alpha \sim \gamma \)

**Cautious Monotonicity:** \( \alpha \land \beta \sim \gamma \)

**propositional logic:** \( \alpha \land \beta \land \gamma \models \beta \land \gamma \)

**Supraclassicality:** \( \alpha \land \beta \land \gamma \sim \beta \land \gamma \)

**Cut:** \( \alpha \land \beta \sim \beta \land \gamma \)

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**MPC is an Exercise.**
### Equivalence

**Assumption:** \(\alpha \equiv \beta, \ \beta \equiv \alpha, \ \alpha \equiv \gamma\)

**Cautious Monotonicity:** \(\alpha \land \beta \equiv \gamma\)

**Left L Equivalence:** \(\beta \land \alpha \equiv \gamma\)

**Cut:** \(\beta \equiv \gamma\)

### And

**Assumption:** \(\alpha \equiv \beta, \ \alpha \equiv \gamma\)

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**Supraclasicality:** \(\alpha \land \beta \land \gamma \equiv \beta \land \gamma\)

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**Cautious Monotonicity:**  \( \alpha \land \beta \not\models \gamma \)

**Left L Equivalence:**  \( \beta \land \alpha \not\models \gamma \)

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### And

**Assumption:**  \( \alpha \not\models \beta, \ \alpha \not\models \gamma \)

**Cautious Monotonicity:**  \( \alpha \land \beta \not\models \gamma \)

**propositional logic:**  \( \alpha \land \beta \land \gamma \models \beta \land \gamma \)

**Supracl assicality:**  \( \alpha \land \beta \land \gamma \not\models \beta \land \gamma \)

**Cut:**  \( \alpha \land \beta \not\models \beta \land \gamma \)

**Cut:**  \( \alpha \not\models \beta \land \gamma \)

**MPC** is an Exercise.
### Equivalence

**Assumption:** \(\alpha \not\models \beta, \quad \beta \not\models \alpha, \quad \alpha \not\models \gamma\)

**Cautious Monotonicity:** \(\alpha \land \beta \not\models \gamma\)

**Left L Equivalence:** \(\beta \land \alpha \not\models \gamma\)

**Cut:** \(\beta \not\models \gamma\)

---

### And

**Assumption:** \(\alpha \not\models \beta, \quad \alpha \not\models \gamma\)

**Cautious Monotonicity:** \(\alpha \land \beta \not\models \gamma\)

**propositional logic:** \(\alpha \land \beta \land \gamma \models \beta \land \gamma\)

**Supraclassicality:** \(\alpha \land \beta \land \gamma \models \beta \land \gamma\)

**Cut:** \(\alpha \land \beta \not\models \beta \land \gamma\)

**Cut:** \(\alpha \not\models \beta \land \gamma\)

**MPC is an Exercise.**
Proofs

**Equivalence**

Assumption: \( \alpha \not\vdash \beta, \ \beta \not\vdash \alpha, \ \alpha \not\vdash \gamma \)

Cautious Monotonicity: \( \alpha \land \beta \not\vdash \gamma \)

Left L Equivalence: \( \beta \land \alpha \not\vdash \gamma \)

Cut: \( \beta \not\vdash \gamma \)

**And**

Assumption: \( \alpha \not\vdash \beta, \ \alpha \not\vdash \gamma \)

Cautious Monotonicity: \( \alpha \land \beta \not\vdash \gamma \)

propositional logic: \( \alpha \land \beta \land \gamma \vdash \beta \land \gamma \)

Supraclassicality: \( \alpha \land \beta \land \gamma \not\vdash \beta \land \gamma \)

Cut: \( \alpha \land \beta \not\vdash \beta \land \gamma \)

Cut: \( \alpha \not\vdash \beta \land \gamma \)

MPC is an Exercise.
Proofs

### Equivalence

**Assumption:** \( \alpha \vdash \neg \beta, \quad \beta \vdash \neg \alpha, \quad \alpha \vdash \neg \gamma \)

**Cautious Monotonicity:** \( \alpha \land \beta \vdash \neg \gamma \)

**Left L Equivalence:** \( \beta \land \alpha \vdash \neg \gamma \)

**Cut:** \( \beta \vdash \neg \gamma \)

### And

**Assumption:** \( \alpha \vdash \neg \beta, \quad \alpha \vdash \neg \gamma \)

**Cautious Monotonicity:** \( \alpha \land \beta \vdash \neg \gamma \)

**propositional logic:** \( \alpha \land \beta \land \gamma \models \beta \land \gamma \)

**Supraclassicality:** \( \alpha \land \beta \land \gamma \vdash \beta \land \gamma \)

**Cut:** \( \alpha \land \beta \vdash \beta \land \gamma \)

**Cut:** \( \alpha \vdash \beta \land \gamma \)

**MPC is an Exercise.**
Proofs

**Equivalence**

- **Assumption:** $\alpha \not\models \beta$, $\beta \not\models \alpha$, $\alpha \not\models \gamma$
- **Cautious Monotonicity:** $\alpha \land \beta \not\models \gamma$
- **Left L Equivalence:** $\beta \land \alpha \not\models \gamma$
- **Cut:** $\beta \not\models \gamma$

**And**

- **Assumption:** $\alpha \not\models \beta$, $\alpha \not\models \gamma$
- **Cautious Monotonicity:** $\alpha \land \beta \not\models \gamma$
- **propositional logic:** $\alpha \land \beta \land \gamma \models \beta \land \gamma$
- **Supraclassicality:** $\alpha \land \beta \land \gamma \not\models \beta \land \gamma$
- **Cut:** $\alpha \land \beta \not\models \beta \land \gamma$
- **Cut:** $\alpha \not\models \beta \land \gamma$

MPC is an Exercise.
Undesirable Properties 1: Monotonicity and Contraposition

- **Monotonicity:**

  \[ \frac{\models \alpha \rightarrow \beta, \beta \not\models \gamma}{\alpha \not\models \gamma} \]

  - **Example:** Let us assume that John goes *normally implies* Mary goes. Now we will probably not expect that John goes *and* Joan (who is not in talking terms with Mary) goes *normally implies* Mary goes.

- **Contraposition:**

  \[ \frac{\alpha \not\models \beta}{\neg \beta \not\models \neg \alpha} \]

  - **Example:** Let us assume that John goes *normally implies* Mary goes. Would we expect that Mary does not go *normally implies* John does not go? What if John goes always?
Undesirable Properties 1: Monotonicity

\[ \alpha \models \beta, \beta \sim \gamma \text{ but not } \alpha \sim \gamma \] pictorially:
Undesirable Properties 1: Contraposition

\( \alpha \sim \beta \) but not \( \neg\beta \sim \neg\alpha \) pictorially:
Undesirable Properties 2: Transitivity & EHD

- **Transitivity:**

\[
\alpha \models \neg \beta, \beta \models \neg \gamma \\
\hline
\alpha \models \neg \gamma
\]

- **Example:** Let us assume that
  
  John goes *normally* implies Mary goes and
  Mary goes *normally* implies Jack goes.

  Now, should John goes *normally imply* that Jack goes?

  If John goes very seldom?

- **Easy Half of Deduction Theorem (EHD):**

\[
\alpha \models \neg \beta \rightarrow \gamma \\
\hline
\alpha \land \beta \models \neg \gamma
\]
Undesirable Properties 2: Transitivity

\[ \alpha \not\sim \beta, \beta \not\sim \gamma \text{ but not } \alpha \not\sim \gamma \text{ pictorially:} \]
Undesirable Properties 2: EHD

\[ \alpha \not\models \beta \rightarrow \gamma \text{ but not } \alpha \land \beta \not\models \gamma \text{ pictorially:} \]
Theorem

In the presence of the rules in system C, monotonicity and EHD are equivalent.

Proof.

Monotonicity $\Rightarrow$ EHD:

- $\alpha \not\vdash \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \not\vdash \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \not\vdash \alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \not\vdash \beta$ (right weakening)
- $\alpha \land \beta \not\vdash \gamma$ (MPC)

Monotonicity $\Leftarrow$ EHD:

- $\models \alpha \rightarrow \beta, \beta \not\vdash \gamma$ (assumption)
- $\beta \not\vdash \alpha \rightarrow \gamma$ (right weakening)
- $\beta \land \alpha \not\vdash \gamma$ (EHD)
- $\alpha \not\vdash \gamma$ (left logical equivalence)
Theorem

In the presence of the rules in system $\text{C}$, monotonicity and EHD are equivalent.

Proof.

$\text{Monotonicity } \Rightarrow \text{ EHD}$:

- $\alpha \not\rightarrow \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \not\rightarrow \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \not\rightarrow \alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \not\rightarrow \beta$ (right weakening)
- $\alpha \land \beta \not\rightarrow \gamma$ (MPC)

$\text{Monotonicity } \Leftarrow \text{ EHD}$:

- $\models \alpha \rightarrow \beta, \beta \not\rightarrow \gamma$ (assumption)
- $\beta \not\rightarrow \alpha \rightarrow \gamma$ (right weakening)
- $\beta \land \alpha \not\rightarrow \gamma$ (EHD)
- $\alpha \not\rightarrow \gamma$ (left logical equivalence)
Theorem

In the presence of the rules in system C, monotonicity and EHD are equivalent.

Proof.

**Monotonicity ⇒ EHD:**

- $\alpha \not\models \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \not\models \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \not\models \alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \not\models \beta$ (right weakening)
- $\alpha \land \beta \not\models \gamma$ (MPC)

**Monotonicity ⇐ EHD:**

- $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
- $\beta \not\models \alpha \rightarrow \gamma$ (right weakening)
- $\beta \land \alpha \not\models \gamma$ (EHD)
- $\alpha \not\models \gamma$ (left logical equivalence)
**Undesirable Properties 3**

**Theorem**

*In the presence of the rules in system C, monotonicity and EHD are equivalent.*

**Proof.**

**Monotonicity $\Rightarrow$ EHD:**

- $\alpha \mid\not\beta \to \gamma$ (assumption)
- $\alpha \land \beta \mid\not\beta \to \gamma$ (monotonicity)
- $\alpha \land \beta \mid\not\alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \mid\not\beta$ (right weakening)
- $\alpha \land \beta \mid\not\gamma$ (MPC)

**Monotonicity $\Leftarrow$ EHD:**

- $\models \alpha \to \beta, \beta \mid\not\gamma$ (assumption)
- $\beta \mid\not\alpha \to \gamma$ (right weakening)
- $\beta \land \alpha \mid\not\gamma$ (EHD)
- $\alpha \mid\not\gamma$ (left logical equivalence)
Theorem

In the presence of the rules in system C, monotonicity and EHD are equivalent.

Proof.

Monotonicity $\Rightarrow$ EHD:

- $\alpha \not\sim \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \not\sim \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \not\sim \alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \not\sim \beta$ (right weakening)
- $\alpha \land \beta \not\sim \gamma$ (MPC)

Monotonicity $\Leftarrow$ EHD:

- $\models \alpha \rightarrow \beta, \beta \not\sim \gamma$ (assumption)
- $\beta \not\sim \alpha \rightarrow \gamma$ (right weakening)
- $\beta \land \alpha \not\sim \gamma$ (EHD)
- $\alpha \not\sim \gamma$ (left logical equivalence)
Theorem

In the presence of the rules in system $\mathbf{C}$, monotonicity and EHD are equivalent.

Proof.

Monotonicity $\Rightarrow$ EHD:

- $\alpha \not\rightarrow \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \not\rightarrow \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \not\rightarrow \alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \not\rightarrow \beta$ (right weakening)
- $\alpha \land \beta \not\rightarrow \gamma$ (MPC)

Monotonicity $\Leftarrow$ EHD:

- $\models \alpha \rightarrow \beta, \beta \not\rightarrow \gamma$ (assumption)
- $\beta \not\rightarrow \alpha \rightarrow \gamma$ (right weakening)
- $\beta \land \alpha \not\rightarrow \gamma$ (EHD)
- $\alpha \not\rightarrow \gamma$ (left logical equivalence)
Theorem

In the presence of the rules in system C, monotonicity and EHD are equivalent.

Proof.

Monotonicity $\Rightarrow$ EHD:

- $\alpha \not\models \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \not\models \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \not\models \alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \not\models \beta$ (right weakening)
- $\alpha \land \beta \not\models \gamma$ (MPC)

Monotonicity $\Leftarrow$ EHD:

- $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
- $\beta \not\models \alpha \rightarrow \gamma$ (right weakening)
- $\beta \land \alpha \not\models \gamma$ (EHD)
- $\alpha \not\models \gamma$ (left logical equivalence)
**Theorem**

*In the presence of the rules in system C, monotonicity and EHD are equivalent.*

**Proof.**

**Monotonicity ⇒ EHD:**

- \( \alpha \not\vdash \beta \rightarrow \gamma \) (assumption)
- \( \alpha \land \beta \not\vdash \beta \rightarrow \gamma \) (monotonicity)
- \( \alpha \land \beta \not\vdash \alpha \land \beta \) (reflexivity)
- \( \alpha \land \beta \not\vdash \beta \) (right weakening)
- \( \alpha \land \beta \not\vdash \gamma \) (MPC)

**Monotonicity ⇐ EHD:**

- \( \models \alpha \rightarrow \beta, \beta \not\vdash \gamma \) (assumption)
- \( \beta \not\vdash \alpha \rightarrow \gamma \) (right weakening)
- \( \beta \land \alpha \not\vdash \gamma \) (EHD)
- \( \alpha \not\vdash \gamma \) (left logical equivalence)
In the presence of the rules in system C, monotonicity and EHD are equivalent.

Proof.

Monotonicity $\Rightarrow$ EHD:
- $\alpha \not\vdash \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \not\vdash \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \not\vdash \alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \not\vdash \beta$ (right weakening)
- $\alpha \land \beta \not\vdash \gamma$ (MPC)

Monotonicity $\Leftarrow$ EHD:
- $\models \alpha \rightarrow \beta, \beta \not\vdash \gamma$ (assumption)
- $\beta \not\vdash \alpha \rightarrow \gamma$ (right weakening)
- $\beta \land \alpha \not\vdash \gamma$ (EHD)
- $\alpha \not\vdash \gamma$ (left logical equivalence)
Undesirable Properties 3

Theorem

*In the presence of the rules in system C, monotonicity and EHD are equivalent.*

Proof.

**Monotonicity $\Rightarrow$ EHD:**

- $\alpha \not\models \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \not\models \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \not\models \alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \not\models \beta$ (right weakening)
- $\alpha \land \beta \not\models \gamma$ (MPC)

**Monotonicity $\Leftarrow$ EHD:**

- $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
- $\beta \not\models \alpha \rightarrow \gamma$ (right weakening)
- $\beta \land \alpha \not\models \gamma$ (EHD)
- $\alpha \not\models \gamma$ (left logical equivalence)
**Theorem**

*In the presence of the rules in system C, monotonicity and EHD are equivalent.*

**Proof.**

**Monotonicity $\Rightarrow$ EHD:**

- $\alpha \not\vdash \beta \rightarrow \gamma$ (assumption)
- $\alpha \land \beta \not\vdash \beta \rightarrow \gamma$ (monotonicity)
- $\alpha \land \beta \not\vdash \alpha \land \beta$ (reflexivity)
- $\alpha \land \beta \not\vdash \beta$ (right weakening)
- $\alpha \land \beta \not\vdash \gamma$ (MPC)

**Monotonicity $\Leftarrow$ EHD:**

- $\models \alpha \rightarrow \beta, \beta \not\vdash \gamma$ (assumption)
- $\beta \not\vdash \alpha \rightarrow \gamma$ (right weakening)
- $\beta \land \alpha \not\vdash \gamma$ (EHD)
- $\alpha \not\vdash \gamma$ (left logical equivalence)
Theorem

In the presence of the rules in system \( C \), monotonicity and EHD are equivalent.

Proof.

Monotonicity \( \Rightarrow \) EHD:

- \( \alpha \sim \beta \rightarrow \gamma \) (assumption)
- \( \alpha \land \beta \sim \beta \rightarrow \gamma \) (monotonicity)
- \( \alpha \land \beta \sim \alpha \land \beta \) (reflexivity)
- \( \alpha \land \beta \sim \beta \) (right weakening)
- \( \alpha \land \beta \sim \gamma \) (MPC)

Monotonicity \( \Leftarrow \) EHD:

- \( \models \alpha \rightarrow \beta, \beta \sim \gamma \) (assumption)
- \( \beta \sim \alpha \rightarrow \gamma \) (right weakening)
- \( \beta \land \alpha \sim \gamma \) (EHD)
- \( \alpha \sim \gamma \) (left logical equivalence)
Theorem

In the presence of the rules in system C, monotonicity and transitivity are equivalent.

Proof.

Monotonicity $\Rightarrow$ transitivity:
- $\alpha \vdash \beta, \beta \vdash \gamma$ (assumption)
- $\alpha \land \beta \vdash \gamma$ (monotonicity)
- $\alpha \vdash \gamma$ (cut)

Monotonicity $\Leftarrow$ transitivity:
- $\models \alpha \rightarrow \beta, \beta \vdash \gamma$ (assumption)
- $\alpha \models \beta$ (deduction theorem)
- $\alpha \vdash \beta$ (supraclassicality)
- $\alpha \vdash \gamma$ (transitivity)
Theorem

In the presence of the rules in system C, monotonicity and transitivity are equivalent.

Proof.

Monotonicity $\Rightarrow$ transitivity:

- $\alpha \not\models \beta, \beta \not\models \gamma$ (assumption)
- $\alpha \land \beta \not\models \gamma$ (monotonicity)
- $\alpha \not\models \gamma$ (cut)

Monotonicity $\Leftarrow$ transitivity:

- $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
- $\alpha \models \beta$ (deduction theorem)
- $\alpha \not\models \beta$ (supra-classicality)
- $\alpha \not\models \gamma$ (transitivity)
Undesirable Properties 4

**Theorem**

*In the presence of the rules in system $C$, monotonicity and transitivity are equivalent.*

**Proof.**

**Monotonicity $\Rightarrow$ transitivity:**
- $\alpha \not\models \beta, \beta \not\models \gamma$ (assumption)
- $\alpha \land \beta \not\models \gamma$ (monotonicity)
- $\alpha \not\models \gamma$ (cut)

**Monotonicity $\iff$ transitivity:**
- $\models \alpha \to \beta, \beta \not\models \gamma$ (assumption)
- $\alpha \models \beta$ (deduction theorem)
- $\alpha \not\models \beta$ (supra-classicality)
- $\alpha \not\models \gamma$ (transitivity)
Theorem

In the presence of the rules in system C, monotonicity and transitivity are equivalent.

Proof.

Monotonicity ⇒ transitivity:
- \( \alpha \not\sim \beta, \beta \not\sim \gamma \) (assumption)
- \( \alpha \land \beta \not\sim \gamma \) (monotonicity)
- \( \alpha \not\sim \gamma \) (cut)

Monotonicity ⇐ transitivity:
- \( \models \alpha \rightarrow \beta, \beta \not\sim \gamma \) (assumption)
- \( \alpha \models \beta \) (deduction theorem)
- \( \alpha \not\sim \beta \) (supraclasiclality)
- \( \alpha \not\sim \gamma \) (transitivity)
Theorem

In the presence of the rules in system C, monotonicity and transitivity are equivalent.

Proof.

Monotonicity $\Rightarrow$ transitivity:
- $\alpha \not\sim \beta, \beta \not\sim \gamma$ (assumption)
- $\alpha \land \beta \not\sim \gamma$ (monotonicity)
- $\alpha \not\sim \gamma$ (cut)

Monotonicity $\Leftarrow$ transitivity:
- $\models \alpha \rightarrow \beta, \beta \not\sim \gamma$ (assumption)
- $\alpha \models \beta$ (deduction theorem)
- $\alpha \not\sim \beta$ (supraclassicality)
- $\alpha \not\sim \gamma$ (transitivity)
Theorem

In the presence of the rules in system C, monotonicity and transitivity are equivalent.

Proof.

Monotonicity $\Rightarrow$ transitivity:

1. $\alpha \not\sim \beta, \beta \not\sim \gamma$ (assumption)
2. $\alpha \land \beta \not\sim \gamma$ (monotonicity)
3. $\alpha \not\sim \gamma$ (cut)

Monotonicity $\iff$ transitivity:

1. $\models \alpha \rightarrow \beta, \beta \not\sim \gamma$ (assumption)
2. $\alpha \models \beta$ (deduction theorem)
3. $\alpha \not\sim \beta$ (supraclassicality)
4. $\alpha \not\sim \gamma$ (transitivity)
In the presence of the rules in system C, monotonicity and transitivity are equivalent.

Proof.

Monotonicity ⇒ transitivity:
\[
\begin{align*}
\alpha &\not\sim \beta, \beta \not\sim \gamma \quad \text{(assumption)} \\
\alpha \land \beta &\not\sim \gamma \quad \text{(monotonicity)} \\
\alpha &\not\sim \gamma \quad \text{(cut)}
\end{align*}
\]

Monotonicity ⇔ transitivity:
\[
\begin{align*}
\vdash \alpha \rightarrow \beta, \beta \not\sim \gamma \quad \text{(assumption)} \\
\alpha &\not\equiv \beta \quad \text{(supra-classicality)} \\
\alpha &\not\sim \gamma \quad \text{(transitivity)}
\end{align*}
\]
Theorem

In the presence of right weakening, contraposition implies monotonicity.

Proof.

1. \( \models \alpha \rightarrow \beta, \beta \not\models \gamma \) (assumption)
2. \( \neg \gamma \not\models \neg \beta \) (contraposition)
3. \( \models \neg \beta \rightarrow \neg \alpha \) (classical contraposition)
4. \( \neg \gamma \not\models \neg \alpha \) (right weakening)
5. \( \alpha \not\models \gamma \) (contraposition)

Note: Monotonicity does not imply contraposition, even in the presence of all rules of system C!
Theorem

In the presence of right weakening, contraposition implies monotonicity.

Proof.

1. \( \models \alpha \rightarrow \beta, \beta \models \sim \gamma \) (assumption)
2. \( \sim \gamma \models \sim \sim \beta \) (contraposition)
3. \( \models \sim \beta \rightarrow \sim \alpha \) (classical contraposition)
4. \( \sim \gamma \models \sim \sim \alpha \) (right weakening)
5. \( \sim \sim \alpha \models \gamma \) (contraposition)

Note: Monotonicity does not imply contraposition, even in the presence of all rules of system C!
Theorem

*In the presence of right weakening, contraposition implies monotonicity.*

Proof.

1. \(\models \alpha \to \beta, \beta \not\models \gamma\) (assumption)
2. \(\neg \gamma \not\models \neg \beta\) (contraposition)
3. \(\models \neg \beta \to \neg \alpha\) (classical contraposition)
4. \(\neg \gamma \not\models \neg \alpha\) (right weakening)
5. \(\alpha \not\models \gamma\) (contraposition)

Note: Monotonicity does not imply contraposition, even in the presence of all rules of system C!
Theorem

In the presence of right weakening, contraposition implies monotonicity.

Proof.

1. $\models \alpha \rightarrow \beta, \beta \not\models \gamma$ (assumption)
2. $\neg \gamma \not\models \neg \beta$ (contraposition)
3. $\models \neg \beta \rightarrow \neg \alpha$ (classical contraposition)
4. $\neg \gamma \not\models \neg \alpha$ (right weakening)
5. $\alpha \not\models \gamma$ (contraposition)

Note: Monotonicity does not imply contraposition, even in the presence of all rules of system C!
Theorem

In the presence of right weakening, contraposition implies monotonicity.

Proof.

1. $\models \alpha \rightarrow \beta, \beta \models \neg \gamma$ (assumption)
2. $\neg \gamma \models \neg \beta$ (contraposition)
3. $\models \neg \beta \rightarrow \neg \alpha$ (classical contraposition)
4. $\neg \gamma \models \neg \alpha$ (right weakening)
5. $\alpha \models \gamma$ (contraposition)

Note: Monotonicity does not imply contraposition, even in the presence of all rules of system C!
Theorem

In the presence of right weakening, contraposition implies monotonicity.

Proof.

1. \( \models \alpha \to \beta, \beta \not\models \gamma \) (assumption)
2. \( \neg \gamma \not\models \neg \beta \) (contraposition)
3. \( \models \neg \beta \to \neg \alpha \) (classical contraposition)
4. \( \neg \gamma \not\models \neg \alpha \) (right weakening)
5. \( \alpha \not\models \gamma \) (contraposition)

Note: Monotonicity does not imply contraposition, even in the presence of all rules of system C!
Theorem

In the presence of right weakening, contraposition implies monotonicity.

Proof.

\[
\begin{align*}
&\ 1 \quad \models \alpha \rightarrow \beta, \beta \not\models \gamma \ (\text{assumption}) \\
&\ 2 \quad \not\gamma \not\models \not\beta \ (\text{contraposition}) \\
&\ 3 \quad \models \not\beta \rightarrow \not\alpha \ (\text{classical contraposition}) \\
&\ 4 \quad \not\gamma \not\models \not\alpha \ (\text{right weakening}) \\
&\ 5 \quad \alpha \not\models \gamma \ (\text{contraposition})
\end{align*}
\]

Note: Monotonicity does not imply contraposition, even in the presence of all rules of system $C$!
Cumulative Closure 1

- How do we reason with $\sim$ from $\varphi$ to $\psi$?
- **Assumption**: We have a set $K$ of conditional statements of the form $\alpha \mid \sim \beta$.
  The question is: Assuming the statements in $K$, is it plausible to conclude $\psi$ given $\varphi$?
- **Idea**: We consider all cumulative consequence relations that contain $K$.
- **Remark**: It suffices to consider only the *minimal* cumulative consequence relations containing $K$. 
Lemma

The set of cumulative consequence relations is closed under intersection.

Proof.

Let $\sim_1$ and $\sim_2$ be cumulative consequence relations. We have to show that $\sim_1 \cap \sim_2$ is a cumulative consequence relation, that is, it satisfies the rules 1–5.

Take any instance of any of the rules. If the preconditions are satisfied by $\sim_1$ and $\sim_2$, then the consequence is trivially also satisfied by both.
Lemma

The set of cumulative consequence relations is closed under intersection.

Proof.

Let $\sim_1$ and $\sim_2$ be cumulative consequence relations. We have to show that $\sim_1 \cap \sim_2$ is a cumulative consequence relation, that is, it satisfies the rules 1–5.

Take any instance of any of the rules. If the preconditions are satisfied by $\sim_1$ and $\sim_2$, then the consequence is trivially also satisfied by both.
Lemma

The set of cumulative consequence relations is closed under intersection.

Proof.

Let $\sim_1$ and $\sim_2$ be cumulative consequence relations. We have to show that $\sim_1 \cap \sim_2$ is a cumulative consequence relation, that is, it satisfies the rules 1–5.

Take any instance of any of the rules. If the preconditions are satisfied by $\sim_1$ and $\sim_2$, then the consequence is trivially also satisfied by both.
Theorem

For each finite set of conditional statements $K$, there exists a unique minimal cumulative consequence relation containing $K$.

Proof.

Assume the contrary, i.e., there are incomparable minimal sets $K_1, \ldots, K_m$. Then $K = K_1 \cap \cdots \cap K_m$ is a unique smallest cumulative consequence relation containing $K$: contradiction.

This relation is the cumulative closure $K^C$ of $K$. 
Theorem

For each finite set of conditional statements $K$, there exists a unique minimal cumulative consequence relation containing $K$.

Proof.

Assume the contrary, i.e., there are incomparable minimal sets $K_1, \ldots, K_m$. Then $K = K_1 \cap \cdots \cap K_m$ is a unique smallest cumulative consequence relation containing $K$: contradiction.

This relation is the cumulative closure $K^C$ of $K$. 
Cumulative Models – informally

- We will now try to characterize cumulative reasoning model-theoretically.
- **Idea:** Cumulative models consist of states ordered by a preference relation.
- **States** characterize beliefs.
- The preference relation expresses the normality of the beliefs.
- We say: $\alpha \models \beta$ is accepted in a model if in all most preferred states in which $\alpha$ is true, also $\beta$ is true.
Let $\prec$ be a binary relation on a set $S$.

$\prec$ is asymmetric iff

\[ s \prec t \implies t \not\prec s \quad \text{for all } s, t \in S. \]

Let $V \subseteq S$ and $\prec$ be a binary relation on $S$.

- $t \in V$ is minimal in $V$ iff $s \not\prec t$ for all $s \in V$.
- $t \in V$ is a minimum of $V$ (a smallest element in $V$) iff $t \prec s$ for all $s \in V$ such that $s \neq t$.

Let $P \subseteq S$ and $\prec$ be a binary relation on $S$.

$P$ is smooth iff for all $t \in P$, either $t$ is minimal in $P$ or there is $s \in P$ such that $s$ is minimal in $P$ and $s \prec t$.

**Note:** $\prec$ is not a partial order but an arbitrary relation!
Let $\mathcal{U}$ be the set of all possible worlds (i.e., propositional interpretations).

- A cumulative model $W$ is a triple $\langle S, l, \prec \rangle$ such that
  1. $S$ is a set of states,
  2. $l$ is a mapping $l : S \rightarrow 2^\mathcal{U}$, and
  3. $\prec$ is an arbitrary binary relation on $S$.

  such that the smoothness condition is satisfied (see below).

- A state $s \in S$ satisfies a formula $\alpha (s \models \alpha)$ iff $m \models \alpha$ for all propositional interpretations $m \in l(s)$.
  The set of states satisfying $\alpha$ is denoted by $\widehat{\alpha}$.

- Smoothness condition: A cumulative model satisfies this condition iff for all formulae $\alpha$, $\widehat{\alpha}$ is smooth.
Consequence Relation Induced by a Cumulative Model

A cumulative model $W$ induces a consequence relation $\sim_W$ as follows:

$$\alpha \sim_W \beta \iff s \models \beta \text{ for every minimal } s \text{ in } \hat{\alpha}.$$ 

**Example**

**Model** $W = \langle \{s_1, s_2, s_3\}, l, \prec \rangle$ with $s_1 \prec s_2, s_2 \prec s_3, s_1 \prec s_3$

<table>
<thead>
<tr>
<th>$l(s_1)$</th>
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Does $W$ satisfy the smoothness condition?

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- Also: $\neg p \land \neg b \not\sim \neg f$!
- $p \sim \neg f$? Y
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Soundness 1

**Theorem**

*If* $W$ *is a cumulative model, then* $\sim_W$ *is a cumulative consequence relation.*

**Proof.**

- **Reflexivity:** satisfied $\checkmark$.
- **Left logical equivalence:** satisfied $\checkmark$.
- **Right weakening:** satisfied $\checkmark$.
- **Cut:** $\alpha \sim \beta$, $\alpha \land \beta \sim \gamma \Rightarrow \alpha \sim \gamma$. Assume that all minimal elements of $\hat{\alpha}$ satisfy $\beta$, and all minimal elements of $\hat{\alpha} \land \hat{\beta}$ satisfy $\gamma$. Every minimal element of $\hat{\alpha}$ satisfies $\alpha \land \beta$. Since $\alpha \land \beta \subseteq \hat{\alpha}$, all minimal elements of $\hat{\alpha}$ are also minimal elements of $\hat{\alpha} \land \hat{\beta}$. Hence $\alpha \sim_W \gamma$. 
Theorem

If \( W \) is a cumulative model, then \( \sim_W \) is a cumulative consequence relation.

Proof.

- **Reflexivity:** satisfied \( \sqrt{\text{.}} \).
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- **Right weakening:** satisfied \( \sqrt{\text{.}} \).
- **Cut:** \( \alpha \sim \beta, \alpha \wedge \beta \sim \gamma \Rightarrow \alpha \sim \gamma \). Assume that all minimal elements of \( \hat{\alpha} \) satisfy \( \beta \), and all minimal elements of \( \alpha \wedge \beta \) satisfy \( \gamma \). Every minimal element of \( \hat{\alpha} \) satisfies \( \alpha \wedge \beta \). Since \( \alpha \wedge \beta \subseteq \hat{\alpha} \), all minimal elements of \( \hat{\alpha} \) are also minimal elements of \( \alpha \wedge \beta \). Hence \( \alpha \sim_W \gamma \).
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Proof continues...

**Cautious Monotonicity:** To show: $\alpha \not\sim \beta$, $\alpha \not\sim \gamma \Rightarrow \alpha \land \beta \not\sim \gamma$.

Assume $\alpha \not\sim_W \beta$ and $\alpha \not\sim_W \gamma$. We have to show:

$\alpha \land \beta \not\sim_W \gamma$, i.e., $s \models \gamma$ for all minimal $s \in \alpha \land \beta$.

Clearly, every minimal $s \in \alpha \land \beta$ is in $\hat{\alpha}$.

We show that every minimal $s \in \alpha \land \beta$ is *minimal* in $\hat{\alpha}$.

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Hence $s$ must be minimal in $\hat{\alpha}$, and therefore $s \models \gamma$. Because this is true for all minimal elements in $\alpha \land \beta$, we get $\alpha \land \beta \not\sim_W \gamma$. 


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Hence \( s \) must be minimal in \( \hat{\alpha} \), and therefore \( s \equiv \gamma \). Because this is true for all minimal elements in \( \hat{\alpha} \land \hat{\beta} \), we get \( \alpha \land \beta \not\sim_W \gamma \).
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Hence \( s \) must be minimal in \( \hat{\alpha} \), and therefore \( s \models \gamma \). Because this is true for all minimal elements in \( \hat{\alpha} \land \hat{\beta} \), we get \( \alpha \land \beta \not\models_{W} \gamma \).
Cautious Monotonicity: To show: $\alpha \models W \beta$, $\alpha \models W \gamma \Rightarrow \alpha \land \beta \models W \gamma$.

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Hence $s$ must be minimal in $\hat{\alpha}$, and therefore $s \equiv \gamma$. Because this is true for all minimal elements in $\alpha \land \beta$, we get $\alpha \land \beta \models W \gamma$. 

Proof continues...
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  Assume $\alpha \not\sim_W \beta$ and $\alpha \not\sim_W \gamma$. We have to show: $\alpha \land \beta \not\sim_W \gamma$, i.e., $s \equiv \gamma$ for all minimal $s \in \hat{\alpha} \land \hat{\beta}$.

  Clearly, every minimal $s \in \hat{\alpha} \land \hat{\beta}$ is in $\hat{\alpha}$.

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  Hence $s$ must be minimal in $\hat{\alpha}$, and therefore $s \equiv \gamma$. Because this is true for all minimal elements in $\hat{\alpha} \land \hat{\beta}$, we get $\alpha \land \beta \not\sim_W \gamma$. 
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  Clearly, every minimal $s \in \hat{\alpha} \land \hat{\beta}$ is in $\hat{\alpha}$.

  We show that every minimal $s \in \hat{\alpha} \land \hat{\beta}$ is *minimal* in $\hat{\alpha}$.

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Now we have a method for showing that a principle does not hold for cumulative consequence relations.

\[ \sim \] Simply construct a cumulative model that falsifies the principle.

**Contraposition:** \( \alpha \sim \beta \Rightarrow \neg \beta \sim \neg \alpha \)

\[
W = \langle S, l, \prec \rangle
\]

\[
S = \{s_1, s_2\}
\]

\[
s_i \not\prec s_j \ \forall s_i, s_j \in S
\]

\[
l(s_1) = \{\{a, b\}\}
\]

\[
l(s_2) = \{\{a, \neg b\}, \{\neg a, \neg b\}\}
\]

\( W \) is a cumulative model with \( a \sim_W b \) but \( \neg b \not\sim_W \neg a \).
Completeness?

- Each cumulative model $W$ *induces* a cumulative consequence relation $\models_W$.
- **Problem**: Can we generate all cumulative consequence relations in this way?
- We can! There is a *representation theorem*: For each cumulative consequence relation, there is a cumulative model and *vice versa*.
- **Advantage**: We have a characterization of the cumulative consequence independently from the set of inference rules.
Transitivity of the Preference Relation?

Could we strengthen the preference relation to transitive relations without sacrificing anything?

No!

In such models, the following additional principle called Loop is valid:

\[
\alpha_0 \sim \alpha_1, \alpha_1 \sim \alpha_2, \ldots, \alpha_k \sim \alpha_0 \quad \Rightarrow \quad \alpha_0 \sim \alpha_k
\]

For the system \( CL = C + \text{Loop} \) and cumulative models with transitive preference relations, we could prove another representation theorem.
The Or Rule

**Or rule:**

\[
\alpha \not\sim \gamma, \beta \not\sim \gamma \\
\frac{}{\alpha \lor \beta \not\sim \gamma}
\]

Not true in C. **Counterexample:**

\[
W = \langle S, l, \prec \rangle \\
S = \{s_1, s_2, s_3\}, s_i \not\prec s_j \ \forall s_i, s_j \in S \\
l(s_1) = \{\{a, b, c\}, \{a, \neg b, c\}\} \\
l(s_2) = \{\{a, b, c\}, \{\neg a, b, c\}\} \\
l(s_3) = \{\{a, b, \neg c\}, \{a, \neg b, \neg c\}, \{\neg a, b, \neg c\}\}
\]

\[a \not\sim_W c, b \not\sim_W c \text{ but } a \lor b \not\not\sim_W c.\]

**Note:** Or is not valid in DL.
System $\mathbf{P}$

- System $\mathbf{P}$ contains all rules of $\mathbf{C}$ and the Or rule.
- A consequence relation that satisfies $\mathbf{P}$ is called preferential.
- Derived rules in $\mathbf{P}$:
  - Hard half of deduction theorem ($S$):
    \[
    \frac{\alpha \land \beta \models \gamma}{\alpha \models \beta \rightarrow \gamma}
    \]
  - Proof by case analysis ($D$):
    \[
    \frac{\alpha \land \neg \beta \models \gamma, \quad \alpha \land \beta \models \gamma}{\alpha \models \gamma}
    \]
- $D$ and Or are equivalent in the presence of the rules in $\mathbf{C}$. 
Definition

A cumulative model $W = \langle S, l, \prec \rangle$ such that $\prec$ is a strict partial order (irreflexive and transitive) and $|l(s)| = 1$ for all $s \in S$ is a preferential model.
Theorem (Soundness)

The consequence relation $\sim_W$ induced by a preferential model is preferential.

Proof.

Since $W$ is cumulative, we only have to verify that Or holds. Note that in preferential models we have $\hat{\alpha} \lor \hat{\beta} = \hat{\alpha} \cup \hat{\beta}$. Suppose $\alpha \sim_W \gamma$ and $\beta \sim_W \gamma$. Because of the above equation, each minimal state of $\hat{\alpha} \lor \hat{\beta}$ is minimal in $\hat{\alpha} \cup \hat{\beta}$. Since $\gamma$ is satisfied in all minimal states in $\hat{\alpha} \cup \hat{\beta}$, $\gamma$ is also satisfied in all minimal states of $\hat{\alpha} \lor \hat{\beta}$. Hence $\alpha \lor \beta \sim_W \gamma$. 

\[\square\]
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[Proof box]
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Theorem (Representation)

A consequence relation is preferential iff it is induced by a preferential model.

Proof.

Similar to the one for C.
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## Summary of Consequence Relations

<table>
<thead>
<tr>
<th>System</th>
<th>Models</th>
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<tbody>
<tr>
<td><strong>C</strong></td>
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<tr>
<td>Reflexivity</td>
<td>States: sets of worlds</td>
</tr>
<tr>
<td>Left Logical Equivalence</td>
<td>Preference relation: arbitrary</td>
</tr>
<tr>
<td>Right Weakening</td>
<td>Models must be smooth</td>
</tr>
<tr>
<td>Cut</td>
<td></td>
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<tr>
<td>Cautious Monotonicity</td>
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<tr>
<th><strong>CL</strong></th>
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<tr>
<td>+ Loop</td>
<td>Preference relation: strict partial order</td>
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<tr>
<th><strong>P</strong></th>
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<tbody>
<tr>
<td>+ Or</td>
<td>States: singletons</td>
</tr>
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</table>
Strengthening the Consequence Relation

- System $\mathbf{C}$ and System $\mathbf{P}$ do not produce many of the inferences one would hope for:

  Given $K = \{\text{Bird} \not\sim \text{Flies}\}$ one cannot conclude $\text{Red} \land \text{Bird} \not\sim \text{Flies}$!

- In general, adding information that is irrelevant cancels the plausible conclusions.
  $\quad\Rightarrow$ Cumulative and Preferential consequence relations are too nonmonotonic.

- The plausible conclusions have to be strengthened!
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In general, adding information that is irrelevant cancels the plausible conclusions.

$\implies$ Cumulative and Preferential consequence relations are too nonmonotonic.

The plausible conclusions have to be strengthened!
The rules so far seem to be reasonable and one cannot think of rules of the same form (if we have some plausible implications, other plausible implications should hold) that could be added.

However, there are other types of rules one might want to add.

- **Disjunctive Rationality:**
  \[
  \alpha \nvdash \gamma, \beta \nvdash \gamma \quad \Rightarrow \quad \alpha \lor \beta \nvdash \gamma
  \]

- **Rational Monotonicity:**
  \[
  \alpha \nmodels \gamma, \alpha \nvdash \neg \beta \quad \Rightarrow \quad \alpha \land \beta \nmodels \gamma
  \]

- **Note:** Consequence relations obeying these rules are not closed under intersection, which is a problem.
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  \not\Vdash \gamma
  \]
  \[
  \alpha \lor \beta 
  \not\Vdash \gamma
  \]

- **Rational Monotonicity:**
  \[
  \alpha \Vdash \sim \gamma , \quad \alpha 
  \not\Vdash \neg \beta
  \]
  \[
  \alpha \land \beta \Vdash \sim \gamma
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  \[
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Probabilistic View of Plausible Consequences

- Consider probability distributions $P$ on the set $\mathcal{M}$ of all propositional interpretations $m \in \mathcal{M}$ of our language.
- $P(m)$ is the probability of the possible world $m$.
- Extend this to probability of formulae:

$$P(\alpha) = \sum \{P(m) | m \in \mathcal{M}, m \models \alpha\}$$

- Conditional probability is defined in the standard way.

$$P(\beta | \alpha) = \frac{P(\alpha \land \beta)}{P(\alpha)}$$
\( \epsilon \)-Entailment

**Definition**

\( \alpha \models \beta \) is \( \epsilon \)-entailed by a set \( K \) iff for each \( \epsilon > 0 \) there is a \( \delta > 0 \) such that for all probability distributions \( P \) with \( P(\beta'|\alpha') \geq 1 - \delta \) for all \( \alpha' \models \beta' \in K \), \( P(\beta|\alpha) \geq 1 - \epsilon \).
**ε-Entailment: Example**

One probability distribution $P$ such that $P(f|b) \geq 0.9$, $P(\neg f|p) \geq 0.9$ and $P(b|p) \geq 0.9$ is the following.

<table>
<thead>
<tr>
<th></th>
<th>$p$</th>
<th>$b$</th>
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<th>$P$</th>
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<td>$w_1$</td>
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\[
P(f|b) = \frac{P(w_4) + P(w_8)}{P(w_3) + P(w_4) + P(w_7) + P(w_8)} = \frac{0.99}{1.00}
\]

\[
P(\neg f|p) = \frac{P(w_5) + P(w_7)}{P(w_5) + P(w_6) + P(w_7) + P(w_8)} = \frac{0.01}{0.01}
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\[
\begin{array}{cc|c|c}
 p & b & f & P \\
 \hline
 w_1 & 0 & 0 & 0 & 0.00 \\
 w_2 & 0 & 0 & 1 & 0.00 \\
 w_3 & 0 & 1 & 0 & 0.00 \\
 w_4 & 0 & 1 & 1 & 0.99 \\
 w_5 & 1 & 0 & 0 & 0.00 \\
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P(f|b) = \frac{P(w_4)+P(w_8)}{P(w_3)+P(w_4)+P(w_7)+P(w_8)} = \frac{0.99}{1.00}
\]

\[
P(\neg f|p) = \frac{P(w_5)+P(w_7)}{P(w_5)+P(w_6)+P(w_7)+P(w_8)} = \frac{0.01}{0.01}
\]

\[
P(b|p) = \frac{P(w_7)+P(w_8)}{P(w_5)+P(w_6)+P(w_7)+P(w_8)} = \frac{0.01}{0.01}
\]
\(\epsilon\)-Entailment: Example

One probability distribution \(P\) such that \(P(f|b) \geq 0.9\), \(P(\neg f|p) \geq 0.9\) and \(P(b|p) \geq 0.9\) is the following.

\[
\begin{array}{c|ccc|c}
 p & b & f & P \\
\hline
 w_1 & 0 & 0 & 0 & 0.00 \\
 w_2 & 0 & 0 & 1 & 0.00 \\
 w_3 & 0 & 1 & 0 & 0.00 \\
 w_4 & 0 & 1 & 1 & 0.99 \\
 w_5 & 1 & 0 & 0 & 0.00 \\
 w_6 & 1 & 0 & 1 & 0.00 \\
 w_7 & 1 & 1 & 0 & 0.01 \\
 w_8 & 1 & 1 & 1 & 0.00 \\
\end{array}
\]

\[
P(f|b) = \frac{P(w_4) + P(w_8)}{P(w_3) + P(w_4) + P(w_7) + P(w_8)} = \frac{0.99}{1.00}
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P(\neg f|p) = \frac{P(w_5) + P(w_7)}{P(w_5) + P(w_6) + P(w_7) + P(w_8)} = \frac{0.01}{0.01}
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\]
Properties of $\epsilon$-Entailment

**Theorem**

$\alpha \not\models \beta$ is in all preferential consequence relations that include $K$ if and only if $\alpha \not\models \beta$ is $\epsilon$-entailed by $K$.

So, System $P$ provides a proof system that exactly corresponds to $\epsilon$-entailment.
Weakness of $\varepsilon$-Entailment

**Question:** Why is Eagle $\sim$ Flies not an $\varepsilon$-consequence of $K = \{\text{Eagle} \sim \text{Bird}, \text{Bird} \sim \text{Flies}\}$?

**Answer:** Because there are probability distributions that simultaneously assign very high probabilities to $P(\text{Bird}|\text{Eagle})$ and $P(\text{Flies}|\text{Bird})$ and a low probability to $P(\text{Flies}|\text{Eagle})$.

$K$ does not justify the low probability of $P(\text{Flies}|\text{Eagle})$: there are exactly as many worlds satisfying Bird $\wedge$ Eagle $\wedge$ Flies and Bird $\wedge$ Eagle $\wedge$ $\neg$Flies, and the worlds satisfying Bird $\wedge$ Flies have a much higher probability than those satisfying Bird $\wedge$ $\neg$Flies. Why should the probabilities for eagles be the other way round?

We would like to restrict to probability distributions that are not biased toward non-flying eagles without a reason.
Entropy of a Probability Distribution

Definition

The entropy of a probability distribution $P$ is

$$H(P) = - \sum_{m \in M} P(m) \log P(m)$$

The probability distribution with the highest entropy is the one that assigns the same probability to every world.
Definition

\( \alpha \sim \beta \) is **ME-entailed** by a set \( K \) iff for all \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( P(\beta|\alpha) \geq 1 - \epsilon \) for the distribution \( P \) that has the maximum entropy among distributions satisfying \( P(\beta'|\alpha') \geq 1 - \delta \) for all \( \alpha' \sim \beta' \in K \).
The distribution $P$ that has the maximum entropy among distributions such that $P(b|e) \geq 0.9$ and $P(f|b) \geq 0.9$ is the following.

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$$P(f|b) = \frac{P(w_4)+P(w_8)}{P(w_3)+P(w_4)+P(w_7)+P(w_8)} = \frac{0.5255}{0.5839} = 0.9000$$

$$P(b|e) = \frac{P(w_7)+P(w_8)}{P(w_5)+P(w_6)+P(w_7)+P(w_8)} = \frac{0.3672}{0.4080} = 0.9000$$

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Entropy of a Probability Distribution: Example

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ME-Entailment: Examples

1. \{\text{Eagle } \not\sim \text{ Bird}, \text{ Bird } \not\sim \text{ Flies}\} \text{ ME-entails Eagle } \not\sim \text{ Flies}

2. \{\text{Penguin } \not\sim \text{ Bird}, \text{ Bird } \not\sim \text{ Flies}, \text{ Penguin } \not\sim \neg\text{ Flies}\}
   \text{ ME-entails Bird } \land \text{ Penguin } \not\sim \neg\text{ Flies}

3. \{\text{Eagle } \not\sim \text{ Bird}\} \text{ ME-entails } \neg\text{ Bird } \not\sim \neg\text{ Eagle}
Instead of *ad hoc* extensions of the logical machinery, analyze the properties of nonmonotonic consequence relations.

Correspondence between rule system and models for System C, and for System P also wrt. a probabilistic semantics.

Irrelevant information poses a problem. Solution approaches: rational monotonicity, maximum entropy.


Literture II

Judea Pearl.  
*Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference,*  
One section on $\epsilon$-semantics and maximum entropy.

Yoav Shoham.  
*Reasoning about Change.*  
Introduces the idea of preferential models.