Principles of Knowledge Representation and Reasoning
Modal Logics

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Motivation for Studying Modal Logics

- Notions like **believing** and **knowing** require a more general semantics than e.g. propositional logic has.
- Some KR formalisms can be understood as (fragments of) a **propositional modal logic**.
- Application 1: spatial representation formalism **RCC8**
- Application 2: **description logics**
- Application 3: reasoning about time
- Application 4: reasoning about actions, strategies, etc.
Motivation for Modal Logics

Often, we want to state something where we have an “embedded proposition”:

- John believes that it is Sunday.
- I know that $2^{10} = 1024$.

Reasoning with embedded propositions:

- John believes that if it is Sunday, then shops are closed.
- John believes that it is Sunday.
- This implies (assuming belief is closed under modus ponens):
  John believes that shops are closed.

How to formalize this?
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Propositional logic + operators □ & ♦ (Box & Diamond):

\[ \varphi \rightarrow \ldots \text{ classical propositional formula} \]

<table>
<thead>
<tr>
<th>□\varphi'</th>
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□ and ♦ have the same operator precedence as \( \neg \).

Some possible readings of □\varphi:

- Necessarily \( \varphi \) (alethic)
- Always \( \varphi \) (temporal)
- \( \varphi \) should be true (deontic)
- Agent A believes \( \varphi \) (doxastic)
- Agent A knows \( \varphi \) (epistemic)

\( \leadsto \) different semantics for different intended readings
Propositional logic + operators $\square$ & $\Diamond$ (*Box & Diamond*):

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\varphi \rightarrow \ldots \text{ classical propositional formula}
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\varphi' & \quad \Diamond \varphi' \quad \text{Diamond}
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$\square$ and $\Diamond$ have the same operator precedence as $\neg$.

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- Always $\varphi$ (temporal)
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![Formula](image.png)

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Could it be possible to define the meaning of □φ truth-functionally, i.e. by referring to the truth value of φ only?

An attempt to interpret necessity truth-functionally:
- If φ is false, then □φ should be false.
- If φ is true, then ...
  - □φ should be true → □ is the identity function
  - □φ should be false → □φ is identical to falsity

Note: There are only 4 different unary Boolean functions \{T, F\} → \{T, F\}. 
Could it be possible to define the meaning of $\square \varphi$ truth-functionally, i.e. by referring to the truth value of $\varphi$ only?

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Truth-Functional Semantics?

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In classical propositional logic, formulae are interpreted over single interpretations and are evaluated to *true* or *false*.

In modal logics one considers *sets* of interpretations: *possible worlds* (physically possible, conceivable, . . .).

Main idea:

- Consider a world (interpretation) \( w \) and a set of worlds \( W \) which are possible with respect to \( w \).
- A classical formula (with no modal operators) \( \varphi \) is true with respect to \( (w, W) \) iff \( \varphi \) is true in \( w \).
- \( \Box \varphi \) is true wrt \( (w, W) \) iff \( \varphi \) is true in all worlds in \( W \).
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- Meanings of \( \Box \) and \( \Diamond \) are inter-related by: \( \Diamond \varphi \equiv \neg \Box \neg \varphi \).
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- $a \land \neg b$ is true relative to $(w, W)$.
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Frames, Interpretations, and Worlds

A (Kripke, relational) frame is a pair \( \mathcal{F} = \langle W, R \rangle \) where \( W \) is a non-empty set (of worlds) and \( R \subseteq W \times W \) (the accessibility relation).

For \((w, v) \in R\) we write also \( wRv \).

We say that \( v \) is an \( R \)-successor of \( w \) and that \( v \) is reachable (or \( R \)-reachable) from \( w \).

A \((\Sigma)\)-interpretation (or model) based on the frame \( \mathcal{F} = \langle W, R \rangle \) is a triple \( \mathcal{I} = \langle W, R, \pi \rangle \), where \( \pi \) is a function from worlds to truth assignments:

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\pi : W \rightarrow (\Sigma \rightarrow \{T, F\}).
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A formula $\varphi$ is **true in world** $w$ of an interpretation $\mathcal{I} = \langle W, R, \pi \rangle$ under the following conditions:

- $\mathcal{I}, w \models a$ iff $\pi(w)(a) = T$
- $\mathcal{I}, w \models \top$
- $\mathcal{I}, w \not\models \bot$
- $\mathcal{I}, w \models \neg \varphi$ iff $\mathcal{I}, w \not\models \varphi$
- $\mathcal{I}, w \models \varphi \land \psi$ iff $\mathcal{I}, w \models \varphi$ and $\mathcal{I}, w \models \psi$
- $\mathcal{I}, w \models \varphi \lor \psi$ iff $\mathcal{I}, w \models \varphi$ or $\mathcal{I}, w \models \psi$
- $\mathcal{I}, w \models \varphi \rightarrow \psi$ iff if $\mathcal{I}, w \models \varphi$ then $\mathcal{I}, w \models \psi$
- $\mathcal{I}, w \models \varphi \leftrightarrow \psi$ iff $\mathcal{I}, w \models \varphi$ if and only if $\mathcal{I}, w \models \psi$
- $\mathcal{I}, w \models \Box \varphi$ iff $\mathcal{I}, u \models \varphi$ for all $u$ s.t. $wRu$
- $\mathcal{I}, w \models \Diamond \varphi$ iff $\mathcal{I}, u \models \varphi$ for at least one $u$ s.t. $wRu$
Semantics: Truth in a World

A formula $\varphi$ is true in world $w$ of an interpretation $\mathcal{I} = \langle W, R, \pi \rangle$ under the following conditions:

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$$\mathcal{I}, w \models \neg \varphi \iff \mathcal{I}, w \not\models \varphi$$

$$\mathcal{I}, w \models \varphi \land \psi \iff \mathcal{I}, w \models \varphi \text{ and } \mathcal{I}, w \models \psi$$

$$\mathcal{I}, w \models \varphi \lor \psi \iff \mathcal{I}, w \models \varphi \text{ or } \mathcal{I}, w \models \psi$$

$$\mathcal{I}, w \models \varphi \rightarrow \psi \iff \text{if } \mathcal{I}, w \models \varphi \text{ then } \mathcal{I}, w \models \psi$$

$$\mathcal{I}, w \models \varphi \leftrightarrow \psi \iff \text{if } \mathcal{I}, w \models \varphi \text{ if and only if } \mathcal{I}, w \models \psi$$

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A formula $\varphi$ is **satisfiable** in an interpretation $I$ (or in a frame $F$, or in a class of frames $C$) if there exists a world in $I$ (or an interpretation $I$ based on $F$, or an interpretation $I$ based on a frame contained in the class $C$, respectively) such that $I,w \models \varphi$.

A formula $\varphi$ is **true** in an interpretation $I$ (symbolically $I \models \varphi$) if $\varphi$ is true in all worlds of $I$.

A formula $\varphi$ is **valid** in a frame $F$ or $F$-valid (symbolically $F \models \varphi$) if $\varphi$ is true in all interpretations based on $F$.

A formula $\varphi$ is **valid** in a class of frames $C$ or $C$-valid (symbolically $C \models \varphi$) if $F \models \varphi$ for all $F \in C$.

$K$ is the class of all frames – named after Saul Kripke, who invented this semantics.
A formula $\varphi$ is **satisfiable** in an interpretation $\mathcal{I}$ (or in a frame $\mathcal{F}$, or in a class of frames $\mathcal{C}$) if there exists a world in $\mathcal{I}$ (or an interpretation $\mathcal{I}$ based on $\mathcal{F}$, or an interpretation $\mathcal{I}$ based on a frame contained in the class $\mathcal{C}$, respectively) such that $\mathcal{I}, w \models \varphi$.

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A formula $\varphi$ is **valid in a class of frames** $\mathcal{C}$ or $\mathcal{C}$-valid (symbolically $\mathcal{C} \models \varphi$) if $\mathcal{F} \models \varphi$ for all $\mathcal{F} \in \mathcal{C}$.

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A formula \( \varphi \) is **satisfiable** in an interpretation \( \mathcal{I} \) (or in a frame \( \mathcal{F} \), or in a class of frames \( \mathcal{C} \)) if there exists a world in \( \mathcal{I} \) (or an interpretation \( \mathcal{I} \) based on \( \mathcal{F} \), or an interpretation \( \mathcal{I} \) based on a frame contained in the class \( \mathcal{C} \), respectively) such that \( \mathcal{I}, w \models \varphi \).

A formula \( \varphi \) is **true** in an interpretation \( \mathcal{I} \) (symbolically \( \mathcal{I} \models \varphi \)) if \( \varphi \) is true in all worlds of \( \mathcal{I} \).

A formula \( \varphi \) is **valid** in a frame \( \mathcal{F} \) or \( \mathcal{F} \)-valid (symbolically \( \mathcal{F} \models \varphi \)) if \( \varphi \) is true in all interpretations based on \( \mathcal{F} \).

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Satisfiability and Validity

A formula $\varphi$ is **satisfiable** in an interpretation $I$ (or in a frame $\mathcal{F}$, or in a class of frames $\mathcal{C}$) if there exists a world in $I$ (or an interpretation $I$ based on $\mathcal{F}$, or an interpretation $I$ based on a frame contained in the class $\mathcal{C}$, respectively) such that $I, w \models \varphi$.

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Validity: Some Examples

1. \( \varphi \lor \neg \varphi \)

2. \( \Box (\varphi \lor \neg \varphi) \)

3. \( \Box \varphi \), if \( \varphi \) is a classical tautology

4. \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \) (axiom schema \( K \))
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Validity: Some Examples

**Theorem**

\[ K \text{ is } K\text{-valid.} \quad (K = \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)) \]

**Proof.**

Let \( \mathcal{I} \) be an interpretation and let \( w \) be a world in \( \mathcal{I} \).

**Assumption:** \( \mathcal{I}, w \models \Box(\varphi \rightarrow \psi) \), i.e., in all worlds \( u \) with \( wRu \), if \( \varphi \) is true then also \( \psi \) is. (Otherwise \( K \) is true in any case.)

If \( \Box \varphi \) is false in \( w \), then \( (\Box \varphi \rightarrow \Box \psi) \) is true and \( K \) is true in \( w \).

If \( \Box \varphi \) is true in \( w \), then both \( \Box(\varphi \rightarrow \psi) \) and \( \Box \varphi \) are true in \( w \).

Hence both \( \varphi \rightarrow \psi \) and \( \varphi \) are true in every world \( u \) accessible from \( w \). Hence \( \psi \) is true in any such \( u \), and therefore \( w \models \Box \psi \). Since \( \mathcal{I} \) and \( w \) were arbitrary, the argument goes through for any \( \mathcal{I}, w \), i.e., \( K \) is \( K \)-valid.
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Validity: Some Examples

**Theorem**

\[ K \text{ is } K\text{-valid.} \quad (K = \square(\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)) \]

**Proof.**

Let \( \mathcal{I} \) be an interpretation and let \( w \) be a world in \( \mathcal{I} \).

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Validitv: Some Examples

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Non-validity: Example

 Proposition

\[ \Diamond \top \textit{ is not } K\text{-valid}. \]

Proof.

A counterexample is the following interpretation:

\[ \mathcal{I} = \langle \{ w \}, \emptyset, \{ w \mapsto (a \mapsto T) \} \rangle. \]

We have \( \mathcal{I}, w \not\models \Diamond \top \) because there is no \( u \) such that \( w Ru \).\]
Non-validity: Example

**Proposition**

\( \Diamond \top \) is not \( \mathbf{K} \)-valid.

**Proof.**

A counterexample is the following interpretation:

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\mathcal{I} = \langle \{w\}, \emptyset, \{w \leftrightarrow (a \leftrightarrow T)\} \rangle.
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Non-validity: Example

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\(\lozenge \top\) is not K-valid.

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Non-validity: Example

Proposition

\(\square \varphi \to \varphi\) is not \(K\)-valid.

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A counterexample is the following interpretation:

\[\mathcal{I} = \langle \{w\}, \emptyset, \{w \mapsto (a \mapsto F)\}\rangle.\]

We have \(\mathcal{I}, w \models \square a\), but \(\mathcal{I}, w \nvdash a\).
Non-validity: Example

Proposition

$\Box \varphi \rightarrow \varphi$ is not K-valid.

Proof.

A counterexample is the following interpretation:

$\mathcal{I} = \langle \{w\}, \emptyset, \{w \mapsto (a \mapsto F')\}\rangle$.

We have $\mathcal{I}, w \models \Box a$, but $\mathcal{I}, w \not\models a$. 

Proof.
Non-validity: Example

**Proposition**

\( \Box \varphi \rightarrow \varphi \) is not \( \mathbf{K} \)-valid.

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A counterexample is the following interpretation:

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We have \( \mathcal{I}, w \models \Box a \), but \( \mathcal{I}, w \not\models a. \)
**Proposition**

\[ \Box \varphi \rightarrow \Box \Box \varphi \text{ is not K-valid.} \]

**Proof.**

A counterexample is the following interpretation:

\[ \mathcal{I} = \langle \{u, v, w\}, \{(u, v), (v, w)\}, \pi \rangle \]

with

\[
\begin{align*}
\pi(u) &= \{a \mapsto T\} \\
\pi(v) &= \{a \mapsto T\} \\
\pi(w) &= \{a \mapsto F\}
\end{align*}
\]

This means \( \mathcal{I}, u \models \Box a \), but \( \mathcal{I}, u \not\models \Box \Box a \).
Non-validity: Another Example

Proposition

\( \Box \varphi \rightarrow \Box \Box \varphi \) is not K-valid.

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Let us consider the following axiom schemata:

- **T**: \( \Box \varphi \rightarrow \varphi \) (knowledge axiom)
- **4**: \( \Box \varphi \rightarrow \Box \Box \varphi \) (positive introspection)
- **5**: \( \Diamond \varphi \rightarrow \Box \Diamond \varphi \) (or \( \neg \Box \varphi \rightarrow \Box \neg \Box \varphi \): negative introspection)
- **B**: \( \varphi \rightarrow \Box \Diamond \varphi \)
- **D**: \( \Box \varphi \rightarrow \Diamond \varphi \) (or \( \Box \varphi \rightarrow \neg \Box \neg \varphi \): disbelief in the negation)

...and the following classes of frames, for which the accessibility relation is restricted as follows:

- **T**: reflexive (\( wRw \) for each world \( w \))
- **4**: transitive (\( wRu \) and \( uRv \) implies \( wRv \))
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Accessibility and Axiom Schemata

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- **D**: serial (for each \( w \) there exists \( v \) with \( w R v \))
Connection between Accessibility Relations and Axiom Schemata (1)

**Theorem**

Axiom schema $T_1 (4, 5, B, D)$ is $T_1$-valid ($4$-, $5$-, $B$-, or $D$-valid, respectively).

**Proof.**

For $T$ and $T_1$: Let $\mathcal{F}$ be a frame from class $T$. Let $\mathcal{I}$ be an interpretation based on $\mathcal{F}$ and let $w$ be an arbitrary world in $\mathcal{I}$. If $\Box \varphi$ is not true in world $w$, then axiom $T$ is true in $w$. If $\Box \varphi$ is true in $w$, then $\varphi$ is true in all accessible worlds. Since the accessibility relation is reflexive, $w$ is among the accessible worlds, i.e., $\varphi$ is true in $w$. This means that also in this case $T$ is true in $w$. This means, $T$ is true in all worlds in all interpretations based on $T_1$-frames.
**Theorem**

Axiom schema $T\ (4, 5, B, D)$ is $T$-valid ($4$-, $5$-, $B$-, or $D$-valid, respectively).

**Proof.**

For $T$ and $T$: Let $\mathcal{F}$ be a frame from class $T$. Let $\mathcal{I}$ be an interpretation based on $\mathcal{F}$ and let $w$ be an arbitrary world in $\mathcal{I}$. If $\Box \varphi$ is not true in world $w$, then axiom $T$ is true in $w$. If $\Box \varphi$ is true in $w$, then $\varphi$ is true in all accessible worlds. Since the accessibility relation is reflexive, $w$ is among the accessible worlds, i.e., $\varphi$ is true in $w$. This means that also in this case $T$ is true in $w$. This means, $T$ is true in all worlds in all interpretations based on $T$-frames.
Connection between Accessibility Relations and Axiom Schemata (1)

Theorem

Axiom schema \( T(4, 5, B, D) \) is \( T \)-valid (4-, 5-, B-, or D-valid, respectively).

Proof.

For \( T \) and \( T \): Let \( \mathcal{F} \) be a frame from class \( T \). Let \( \mathcal{I} \) be an interpretation based on \( \mathcal{F} \) and let \( w \) be an arbitrary world in \( \mathcal{I} \). If \( \square \varphi \) is not true in world \( w \), then axiom \( T \) is true in \( w \). If \( \square \varphi \) is true in \( w \), then \( \varphi \) is true in all accessible worlds. Since the accessibility relation is reflexive, \( w \) is among the accessible worlds, i.e., \( \varphi \) is true in \( w \). This means that also in this case \( T \) is true in \( w \). This means, \( T \) is true in all worlds in all interpretations based on \( T \)-frames.
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For $T$ and $\text{T}$: Let $F$ be a frame from class $\text{T}$. Let $I$ be an interpretation based on $F$ and let $w$ be an arbitrary world in $I$. If $\Box \varphi$ is not true in world $w$, then axiom $T$ is true in $w$. If $\Box \varphi$ is true in $w$, then $\varphi$ is true in all accessible worlds. Since the accessibility relation is reflexive, $w$ is among the accessible worlds, i.e., $\varphi$ is true in $w$. This means that also in this case $T$ is true in $w$. This means, $T$ is true in all worlds in all interpretations based on $\text{T}$-frames.
Theorem

Axiom schema $T(4,5,B,D)$ is $T$-valid ($4$-, $5$-, $B$-, or $D$-valid, respectively).

Proof.

For $T$ and $T$: Let $\mathcal{F}$ be a frame from class $T$. Let $\mathcal{I}$ be an interpretation based on $\mathcal{F}$ and let $w$ be an arbitrary world in $\mathcal{I}$. If $\Box \varphi$ is not true in world $w$, then axiom $T$ is true in $w$. If $\Box \varphi$ is true in $w$, then $\varphi$ is true in all accessible worlds. Since the accessibility relation is reflexive, $w$ is among the accessible worlds, i.e., $\varphi$ is true in $w$. This means that also in this case $T$ is true in $w$. This means, $T$ is true in all worlds in all interpretations based on $T$-frames.
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If $T (4, 5, B, D)$ is valid in a frame $\mathcal{F}$, then $\mathcal{F}$ is a $T$-Frame (4-, 5-, B-, or D-frame, respectively).

Proof.

For $T$ and $T$: Assume that $\mathcal{F}$ is not a $T$-frame. We will construct an interpretation based on $\mathcal{F}$ that falsifies $T$.

Because $\mathcal{F}$ is not a $T$-frame, there is a world $w$ such that not $wRw$.

Construct an interpretation $\mathcal{I}$ such that $w \not\models p$ and $v \models p$ for all $v$ such that $wRv$.

Now $w \models \Box p$ and $w \not\models p$, and hence $w \not\models \Box p \rightarrow p$. □
Connection between Accessibility Relations and Axiom Schemata (2)

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*If* $T(4, 5, B, D)$ *is valid in a frame* $\mathcal{F}$, *then* $\mathcal{F}$ *is a* $T$-*Frame (4-, 5-, B-, or D-frame, respectively).*

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### Different Modal Logics

<table>
<thead>
<tr>
<th>Name</th>
<th>Property</th>
<th>Axiom schema</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K$</td>
<td>—</td>
<td>$\square (\varphi \rightarrow \psi) \rightarrow (\square \varphi \rightarrow \square \psi)$</td>
</tr>
<tr>
<td>$T$</td>
<td>reflexivity</td>
<td>$\square \varphi \rightarrow \varphi$</td>
</tr>
<tr>
<td>4</td>
<td>transitivity</td>
<td>$\square \varphi \rightarrow \square \square \varphi$</td>
</tr>
<tr>
<td>5</td>
<td>euclidicity</td>
<td>$\diamond \varphi \rightarrow \square \diamond \varphi$</td>
</tr>
<tr>
<td>$B$</td>
<td>symmetry</td>
<td>$\varphi \rightarrow \square \diamond \varphi$</td>
</tr>
<tr>
<td>$D$</td>
<td>seriality</td>
<td>$\square \varphi \rightarrow \diamond \varphi$</td>
</tr>
</tbody>
</table>

Some basic modal logics:

\[
K \\
KT4 \quad = \quad S4 \\
KT5 \quad = \quad S5 \\
\vdots
\]
## Different Modal Logics

<table>
<thead>
<tr>
<th>logics</th>
<th>□</th>
<th>♦ = ¬□¬</th>
<th>K</th>
<th>T</th>
<th>4</th>
<th>5</th>
<th>B</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>alethic</td>
<td>necessarily</td>
<td>possibly</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>epistemic</td>
<td>known</td>
<td>possible</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>doxastic</td>
<td>believed</td>
<td>possible</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>deontic</td>
<td>obligatory</td>
<td>permitted</td>
<td>Y</td>
<td>N</td>
<td>Y?</td>
<td>Y?</td>
<td>N</td>
<td>Y</td>
</tr>
<tr>
<td>temporal</td>
<td>always in the future</td>
<td>sometimes</td>
<td>Y</td>
<td>Y/N</td>
<td>Y</td>
<td>N</td>
<td>N</td>
<td>N/Y</td>
</tr>
</tbody>
</table>
How can we show that a formula is $C$-valid?

In order to show that a formula is not $C$-valid, one can construct a counterexample (= an interpretation that falsifies it).

When trying out all ways of generating a counterexample without success, this counts as a proof of validity.

Method of (analytic/semantic) tableaux
Proof Methods

- How can we show that a formula is $C$-valid?
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⇝ Method of (analytic/semantic) tableaux
A tableau is a tree with nodes marked as follows:

- \( w \models \varphi \),
- \( w \not\models \varphi \), and
- \( wRv \).

A branch that contains nodes marked with \( w \models \varphi \) and \( w \not\models \varphi \) is closed. All other branches are open. If all branches are closed, the tableau is called closed.

A tableau is constructed by using the tableau rules.
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## Tableau Rules for the Propositional Logic

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<thead>
<tr>
<th>Rule</th>
<th>Symbol</th>
<th>Conditions</th>
<th>Conclusion</th>
</tr>
</thead>
</table>
| $w \models \varphi \lor \psi$ | $\lor$ | $w \models \varphi$  
$w \models \psi$ | $w \models \varphi \lor \psi$ |
| $w \not\models \varphi \lor \psi$ | $\lor$ | $w \not\models \varphi$  
$w \not\models \psi$ | $w \not\models \varphi \lor \psi$ |
| $w \not\models \neg \varphi$ | $\neg$ | $w \models \varphi$ | $w \not\models \varphi$ |

<table>
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</table>
| $w \models \varphi \land \psi$ | $\land$ | $w \models \varphi$  
$w \models \psi$ | $w \models \varphi \land \psi$ |
| $w \not\models \varphi \land \psi$ | $\land$ | $w \not\models \varphi$  
$w \not\models \psi$ | $w \not\models \varphi \land \psi$ |
| $w \not\models \neg \varphi$ | $\neg$ | $w \not\models \varphi$ | $w \models \neg \varphi$ |

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$w \not\models \psi$ | $w \not\models \varphi \rightarrow \psi$ |

### Tableau Rules

- **Syntax**
- **Semantics**
- **Different logics**
- **Analytic Tableaux**
- **Logical Consequence**
- **Embedding in FOL**
- **Outlook & Literature**
Additional Tableau Rules for the Modal Logic $\textbf{K}$

\[
\frac{w \models \Box \varphi}{v \models \varphi} \quad \text{if } wRv \text{ is on the branch already}
\]

\[
\frac{w \not\models \Box \varphi}{\text{for new } v}
\]

\[
\frac{w \models \Diamond \varphi}{wRv}
\]

\[
\frac{v \models \varphi}{w \not\models \Diamond \varphi} \quad \text{if } wRv \text{ is on the branch already}
\]

\[
\frac{v \not\models \varphi}{w \not\models \Diamond \varphi}
\]
Properties of K Tableaux

Proposition

If a K-tableau is closed, the truth condition at the root cannot be satisfied.

Theorem (Soundness)

If a K-tableau with root \( w \models \varphi \) is closed, then \( \varphi \) is K-valid.

Theorem (Completeness)

If \( \varphi \) is K-valid, then there is a closed tableau with root \( w \models \varphi \).

Proposition (Termination)

There are strategies for constructing K-tableaux that always terminate after a finite number of steps, and result in a closed tableau whenever one exists.
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**Proposition**

*If a $K$-tableau is closed, the truth condition at the root cannot be satisfied.*

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*If a $K$-tableau with root $w \not\models \varphi$ is closed, then $\varphi$ is $K$-valid.*

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Proofs within more restricted classes of frames allow the use of further tableau rules.

- For reflexive (T) frames we may extend any branch with $wRw$.
- For transitive (4) frames we have the following additional rule:
  - If $wRv$ and $vRu$ are in a branch, $wRu$ may be added to the branch.
- For serial (D) frames we have the following rule:
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- Similar rules for other properties...
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- Similar rules for other properties...
Let $\Theta$ be a set of formulas. When does a formula $\varphi$ follow from $\Theta$: $\Theta \models X \varphi$?

Test whether in all interpretations on $X$-frames in which $\Theta$ is true, also $\varphi$ is true.

Wouldn’t there be a deduction theorem we could use?

Example: $a \models K \Box a$ holds, but $a \rightarrow \Box a$ is not $K$-valid.

There is no deduction theorem as in the propositional logic, and logical consequence cannot be directly reduced to validity!
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Tableaux and Logical Implication

For testing logical consequence, we can use the following tableau rule:

- If \( w \) is a world on a branch and \( \psi \in \Theta \), then we can add \( w \models \psi \) to our branch.

- Soundness is obvious.

- Completeness is non-trivial.
Tableaux and Logical Implication

For testing logical consequence, we can use the following tableau rule:

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- Soundness is obvious.
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Connection between propositional modal logic and FOL?

- There are similarities between the predicate logic and propositional modal logics:
  1. $\Box$ vs. $\forall$
  2. $\Diamond$ vs. $\exists$
  3. the possible worlds vs. the objects of the universe
- In fact, we can show for many propositional modal logics that they can be embedded in the predicate logic.
  $\Rightarrow$ Modal logics can be understood as a sublanguage of FOL.
Embedding Modal Logics in the Predicate Logic (1)

\[ \tau(p, x) = p(x) \text{ for propositional variables } p \]

\[ \tau(\neg \phi, x) = \neg \tau(\phi, x) \]

\[ \tau(\phi \lor \psi, x) = \tau(\phi, x) \lor \tau(\psi, x) \]

\[ \tau(\phi \land \psi, x) = \tau(\phi, x) \land \tau(\psi, x) \]

\[ \tau(\Box \phi, x) = \forall y (R(x, y) \rightarrow \tau(\phi, y)) \text{ for some new } y \]

\[ \tau(\Diamond \phi, x) = \exists y (R(x, y) \land \tau(\phi, y)) \text{ for some new } y \]
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4. \( \tau(\phi \land \psi, x) = \tau(\phi, x) \land \tau(\psi, x) \)
5. \( \tau(\Box \phi, x) = \forall y (R(x, y) \rightarrow \tau(\phi, y)) \) for some new \( y \)
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\end{enumerate}
Embedding Modal Logics in the Predicate Logic (2)

**Theorem**

\( \phi \) is K-valid if and only if \( \forall x \tau(\phi, x) \) is valid in the predicate logic.

**Theorem**

\( \phi \) is T-valid if and only if in the predicate logic the logical consequence \( \{\forall x R(x, x)\} \models \forall x \tau(\phi, x) \) holds.

**Example**

\( \square p \land \Diamond (p \rightarrow q) \rightarrow \Diamond q \) is K-valid, because

\[
\forall x (\forall x' (R(x, x') \rightarrow p(x'))) \land \exists x' (R(x, x') \land (p(x') \rightarrow q(x'))) \rightarrow \exists x' (R(x, x') \land q(x'))
\]

is valid in the predicate logic.
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is valid in the predicate logic.
Theorem

φ is K-valid if and only if ∀x τ(φ, x) is valid in the predicate logic.

Theorem

φ is T-valid if and only if in the predicate logic the logical consequence {∀x R(x, x)} |= ∀x τ(φ, x) holds.

Example

□p ∨ (p → q) → ◊q is K-valid, because

∀x(∀x′(R(x, x′) → p(x′)) ∧ ∃x′(R(x, x′) ∧ (p(x′) → q(x′))))
→ ∃x′(R(x, x′) ∧ q(x′)))

is valid in the predicate logic.
We only looked at some basic propositional modal logics. There are also:

- modal first order logics (with quantification $\forall$ and $\exists$ and predicates)
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Outlook

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