Principles of Knowledge Representation and Reasoning

Complexity Theory

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Motivation for Using Complexity Theory

- Complexity theory can answer questions on how easy or hard a problem is.
- Gives hints on what algorithms could be appropriate, e.g.:
  - Algorithms for polynomial-time problems are usually easy to design.
  - For NP-complete problems, backtracking and local search work well.
- Gives hints on what type of algorithm will (most probably) not work.
  - For problems that are believed to be harder than NP-complete ones, simple backtracking will not work.
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This is justified, because:

- we assume that Turing machines can compute all computable functions
- the resource requirements (in terms of time and memory) of a Turing machine are only polynomially worse than other models

The regular type of Turing machine is the **deterministic** one: **DTM** (or simply **TM**)

Often, however, we use the notion of **nondeterministic** TMs: **NDTM**

**Algorithms and Turing Machines**

- Reminder: Basic Notions
- Problems, Solutions, and Complexity
- Complexity Classes P and NP
- Upper and Lower Bounds
- Polynomial Reductions
- NP-Completeness
- Oracle TMs and the Polynomial Hierarchy
- Literature
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A problem is a set of pairs \((I, A)\) of strings in \(\{0, 1\}^*\).
- \(I\): Instance; \(A\): Answer.
- If \(A \in \{0, 1\}\): decision problem

A decision problem is the same as a formal language: namely the set of strings formed by the instances with answer 1.

An algorithm decides (or solves) a problem if it computes the right answer for all instances.

The complexity of an algorithm is a function

\[ T : \mathbb{N} \to \mathbb{N}, \]

measuring the number of basic steps (or memory requirement) the algorithm needs to compute an answer depending on the size of the instance.

The complexity of a problem is the complexity of the most efficient algorithm that solves this problem.
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Complexity Classes P and NP

Problems are categorized into complexity classes according to the requirements of computational resources:

- The class of problems decidable on deterministic Turing machines in polynomial time: $\text{P}$
- Problems in P are assumed to be efficiently solvable (although this might not be true if the exponent is very large)
- In practice, this notion appears to be more often reasonable than not
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Upper and Lower Bounds

- **Upper bounds** (membership in a class) are usually easy to prove:
  - provide an algorithm
  - show that the resource bounds are respected

- **Lower bounds** (hardness for a class) are usually difficult to show:
  - the technical tool here is the polynomial reduction (or any other appropriate reduction)
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Polynomial Reductions

- Given two languages $L_1$ and $L_2$, $L_1$ can be polynomially reduced to $L_2$, written $L_1 \leq_p L_2$, iff there exists a polynomially computable function $f$ such that
  \[ x \in L_1 \text{ iff } f(x) \in L_2 \]

- It cannot be harder to decide $L_1$ than $L_2$

- $L$ is hard for a class $C$ ($C$-hard) iff all languages of this class can be reduced to $L$.

- $L$ is complete for $C$ ($C$-complete) iff $L$ is $C$-hard and $L \in C$. 
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- A problem is **NP-complete** iff it is **NP-hard** and in **NP**.
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Note that there is some asymmetry in the definition of NP:
- It is clear that we can decide SAT by using a NDTM with polynomially bounded computation.
- There exists an accepting computation of polynomial length iff the formula is satisfiable.
- What if we want to solve UNSAT, the complementary problem?
- It seems necessary to check all possible truth-assignments!

Define $\text{co-}C = \{ L | \Sigma^* - L \in C \}$, provided $\Sigma$ is our alphabet.
\[ \text{co-NP} = \{ L | \Sigma^* - L \in \text{NP} \} \]
For example UNSAT, TAUT $\in$ co-NP!

Note: P is closed under complement, i.e.,
\[ P \subseteq \text{NP} \cap \text{co-NP} \]
There are problems even more difficult than NP and co-NP.

**Definition ((N)PSPACE)**

PSPACE (NPSPACE) is the class of decision problems that can be decided on deterministic (non-deterministic) Turing machines using only polynomially many tape cells.

Some facts about PSPACE:

- PSPACE is closed under complements (as all other deterministic classes)
- PSPACE is identical to NPSPACE (because non-deterministic Turing machines can be simulated on deterministic TMs using only quadratic space)
- $\text{NP} \subseteq \text{PSPACE}$ (because in polynomial time one can “visit” only polynomial space, i.e., $\text{NP} \subseteq \text{NPSPACE}$)
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Definition (PSPACE-completeness)

A decision problem (or language) is **PSPACE-complete**, if it is in PSPACE and all other problems in PSPACE can be polynomially reduced to it.

Intuitively, PSPACE-complete problems are the “hardest” problems in PSPACE (similar to NP-completeness). They appear to be “harder” than NP-complete problems from a practical point of view.

An example for a PSPACE-complete problem is the NDFA equivalence problem:

**Instance:** Two non-deterministic finite state automata $A_1$ and $A_2$.

**Question:** Are the languages accepted by $A_1$ and $A_2$ identical?
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- There are complexity classes **above PSPACE** (EXPTIME, EXPSPACE, NEXPTIME, DEXPTIME . . .)

- there are (infinitely many) classes **between NP and PSPACE** (the polynomial hierarchy defined by oracle machines)

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An **Oracle Turing machine** \((N)OTM\) is a Turing machine (DTM, NDTM) with the possibility to query an oracle (i.e., a different Turing machine **without resource restrictions**) whether it accepts or rejects a given string.

- **Computation by the oracle does not cost anything!**
- **Formalization:**
  - a tape onto which strings for the oracle are written,
  - a yes/no answer from the oracle depending on whether it accepts or rejects the input string.
- **Usage of OTMs answers what-if questions:** What if we could solve the oracle-problem efficiently?
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OTMs allow us to define a more general type of reduction

Idea: The “classical” reduction can be seen as calling a subroutine once.

$L_1$ is Turing-reducible to $L_2$, symbolically $L_1 \leq_T L_2$, if there exists a poly-time OTM that decides $L_1$ by using an oracle for $L_2$.

Polynomial reducibility implies Turing reducibility, but not vice versa!

NP-hardness and co-NP-hardness with respect to Turing reducibility are equivalent!

Turing reducibility can also be applied to general search problems!
Turing Reductions

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- **Idea**: The “classical” reduction can be seen as calling a subroutine once.

$L_1$ is Turing-reducible to $L_2$, symbolically $L_1 \leq_T L_2$, if there exists a poly-time OTM that decides $L_1$ by using an oracle for $L_2$.

- Polynomial reducibility implies Turing reducibility, but not vice versa!
- NP-hardness and co-NP-hardness with respect to Turing reducibility are equivalent!
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**Motivation**

Reminder: Basic Notions

Beyond NP

Oracle TMs and the Polynomial Hierarchy

Oracle Turing-Machines

Turing Reduction

Complexity Classes Based on OTMs

QBF

**Literature**

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Complexity Classes Based on Oracle TMs

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- Consider the **Minimum Equivalent Expression (MEE)** problem:

  **Instance:** A well-formed Boolean formula $\phi$ using the standard connectives (not $\leftrightarrow$) and a nonnegative integer $K$.
  **Question:** Is there a well-formed Boolean formula $\phi'$ that contains $K$ or fewer literal occurrences and that is logical equivalent to $\phi$?

- This problem is NP-hard (wrt. to Turing reductions).
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The Polynomial Hierarchy

The complexity classes based on OTMs form an infinite hierarchy.

### The polynomial hierarchy PH

<table>
<thead>
<tr>
<th>Class</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma^p_0$</td>
<td>$P$</td>
</tr>
<tr>
<td>$\Sigma^p_{i+1}$</td>
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Quantified Boolean Formulae: Definition

- If $\phi$ is a propositional formula, $P$ is the set of Boolean variables used in $\phi$ and $\sigma$ is a sequence of $\exists p$ and $\forall p$, one for every $p \in P$, then $\sigma \phi$ is a QBF.

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The evaluation problem of QBF generalizes both the *satisfiability* and *validity/tautology problems* of propositional logic.

The latter are respectively *NP-complete* and *co-NP-complete* whereas the former is *PSPACE-complete*.

**Example**

The formulae $\forall x \exists y (x \leftrightarrow y)$ and $\exists x \exists y (x \land y)$ are true.

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The formulae $\exists x \forall y (x \leftrightarrow y)$ and $\forall x \forall y (x \lor y)$ are false.
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Truth of QBFs with prefix $\forall \exists \forall \ldots$ is $\Pi^p_i$-complete.

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