Principles of Knowledge Representation and Reasoning

Semantic Networks and Description Logics I: Simple, Strict Inheritance Networks

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Simple, Strict Inheritance Networks – Outline

1. Intuition
2. A simple network formalism
3. Semantic Networks with Instances
4. Semantic Networks with Negation
5. Semantic Networks with Negation and Conjunction
A **strict inheritance network** contains **nodes** (concepts, properties) and **directed edges** (generalization/ISA relation).

- **Reasoning problem**: Is a concept $B$ a **specialization** (a subconcept) of another concept $B'$?
- **Question**: Can we solve this problem **efficiently**?
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**Question**: Can we solve this problem **efficiently**?
Networks as Formula Sets

A strict inheritance network is a set $\Theta$ of formulas of the form

$$C_1 \text{ isa } C_2.$$ 

**Example:**

\begin{align*}
\text{Student} & \text{ isa } \text{ Person} \\
\text{Student} & \text{ isa } \text{ studious} \\
\text{Professor} & \text{ isa } \text{ Person} \\
\text{Professor} & \text{ isa } \text{ knowledgeable} \\
\text{Grad-Student} & \text{ isa } \text{ Student} \\
\text{Undergrad-Student} & \text{ isa } \text{ Student}
\end{align*}

**Reasoning Problem (Inheritance):** $\Theta \models C_1 \text{ isa } C_2$. 
A strict inheritance network is a set $\Theta$ of formulas of the form

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**Example:**

- Student isa Person
- Student isa studious
- Professor isa Person
- Professor isa knowledgeable
- Grad-Student isa Student
- Undergrad-Student isa Student

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A strict inheritance network is a set $\Theta$ of formulas of the form

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$$\Theta \models C_1 \text{ isa } C_2.$$
A strict inheritance network is a set \( \Theta \) of formulas of the form

\[
C_1 \text{ isa } C_2.
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**Example:**

- Student \text{ isa } Person
- Student \text{ isa } studious
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**Reasoning Problem (Inheritance):** \( \Theta \models C_1 \text{ isa } C_2. \)
Logical Semantics

- We assign the following logical semantics to `isa`-formulas:

\[ C_1 \text{ isa } C_2 \quad \rightarrow \quad \forall x : C_1(x) \rightarrow C_2(x). \]

- We interpret each directed edge or each `isa`-formula as a universally quantified implication.
- Conforms with intuition: Each instance of a sub-concept is an instance of the super-concept.
- Now we can `reduce` the inheritance problem as follows.
- Let \( \pi(\Theta) \) be the translation. Then we want to know:

\[ \pi(\Theta) \models \forall x : C_1(x) \rightarrow C_2(x). \]

- How hard is this problem?
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- Now we can **reduce** the **inheritance problem** as follows.

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\[ \pi(\Theta) \models \forall x: C_1(x) \rightarrow C_2(x). \]

- How hard is this problem?
Logical Semantics

- We assign the following logical semantics to isa-formulas:

\[ C_1 \text{ isa } C_2 \rightarrow \forall x: C_1(x) \rightarrow C_2(x). \]

- We interpret each directed edge or each isa-formula as a universally quantified implication.
- Conforms with intuition: Each instance of a sub-concept is an instance of the super-concept.
- Now we can reduce the inheritance problem as follows.
- Let \( \pi(\Theta) \) be the translation. Then we want to know:

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A Polynomial Reasoning Algorithm

Let $G_{\Theta}$ be the “graph corresponding to $\Theta$”. Then we have:

$$\pi(\Theta) \models \forall x : C_1(x) \rightarrow C_2(x)$$

iff

there exists a path in $G_{\Theta}$ from $C_1$ to $C_2$.

- ... which has to be proven
- We have reduced reasoning in strict inheritance networks to graph reachability problem, which is solvable in poly. time
- **Note**: Reasoning is not simple because we used a graph to represent the knowledge (there are actually very difficult graph problems).
- Reasoning is simple because the expressiveness compared with first-order logic is very restricted.
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Theorem (Soundness of inheritance reasoning)

(Soundness) If there is a path from $C_1$ to $C_2$ in $G_\Theta$ then
\[ \pi(\Theta) \models \forall x : C_1(x) \rightarrow C_2(x). \]

Proof.
If there is a path, then there exists a chain of implications of the kind $\forall x : D_j(x) \rightarrow D_{j+1}(x)$ with $D_0 = C_1$ and $D_n = C_2$. Since implication is transitive, the claim follows.
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Completeness

**Theorem (Completeness of inheritance reasoning)**

If $\pi(\Theta) \models \forall x: C_1(x) \rightarrow C_2(x)$ then there is a path from $C_1$ to $C_2$ in $G_\Theta$.

**Proof.**

We prove the contraposition by constructing a counter example. Suppose the universe has exactly one element $d$, which is in the interpretation of $C_1$ and in the interpretation of all concepts reachable from $C_1$ by following the directed edges. This interpretation satisfies all formulas in $\pi(\Theta)$. However, it does not satisfy $\forall x: C_1(x) \rightarrow C_2(x)$. For this reason, we have $\pi(\Theta) \not\models \forall x: C_1(x) \rightarrow C_2(x)$. 

$\square$
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An Extension: Instances

We want to talk about **instances** of concepts.

**Example:**

```
studious

Person

knowledgeable

Student

Professor

Undergrad-Student

Grad-Student

John

Bernhard
```

**As formulas:**

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John inst-of Undergrad-Student
Bernhard inst-of Professor
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**Example:**

![Graph showing relationships between concepts and instances]

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Extension of the Semantics

Logical Semantics

\[ i \text{ inst-of } C \leftrightarrow C(i). \]

- **1st Problem**: Is our extension conservative? I.e., can we decide \( \Theta \models C_1 \text{ isa } C_2 \) without taking the formulas \( i \text{ inst-of } C \) into account?
  - yes (has to be shown)

- **2nd Problem**: Is it true: \( \Theta \models i \text{ inst-of } C \) iff there is a path from \( i \) to \( C \) in \( G_\Theta \)?
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I.e., we can use our efficient algorithm for this extension.
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Another Extension: Negated Concepts

We now allow at all places where we had a concept before the expression

\[ \text{not } C, \]

where \( C \) is a concept.

Example:

\[
\text{Undergrad-Student} \quad \text{isa} \quad (\text{not Grad-Student})
\]

Logical semantics:

\[
(\text{not } C) \leftrightarrow \neg C(x).
\]

Example:

\[
C_1 \text{ isa } (\text{not } C_2) \leftrightarrow \forall x: \; C_1(x) \rightarrow \neg C_2(x).
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\[ \text{Undergrad-Student isa (not Grad-Student)} \]

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Completing an Inheritance Network

Define $\bar{\alpha}$:

$$\overline{C} = \text{not } C$$

$$\overline{\text{not } C'} = C$$

Construct $G_\Theta$ from $\Theta$ as follows:

- For each concept name $C$, we will have two nodes: $C$ and $\text{not } C$.
- For each formula $\alpha_1 \text{ isa } \alpha_2$, we introduce the following two edges:

  $$\alpha_1 \rightarrow \alpha_2$$
  $$\overline{\alpha_2} \rightarrow \overline{\alpha_1}$$
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  \alpha_1 \quad \longrightarrow \quad \alpha_2 \\
  \bar{\alpha}_2 \quad \longrightarrow \quad \bar{\alpha}_1
  \]
$\Theta = \{ \text{A isa (not B), P isa A, P isa B,} $
\text{Q isa R, R isa (not A)} \}$
Satisfiability of an Inheritance Network

- Strict inheritance networks without negation are always satisfiable, i.e., they have a non-empty model (which one?)
- This is not true any longer:

\[ P \text{ isa not } P, \text{ not } P \text{ isa } P \]

means

\[ \forall x : P(x) \rightarrow \neg P(x), \forall x : \neg P(x) \rightarrow P(x), \]

which is equivalent to

\[ \forall x : \neg P(x), \forall x : P(x). \]

- The set of formulas is not satisfiable, symbolically \( \Theta \models \).
- This is important to find out since in this case everything follows.
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Satisfiability of an Inheritance Network

- Strict inheritance networks **without negation** are always satisfiable, i.e., they have a non-empty model (which one?)
- This is not true any longer:

\[ P \text{ isa not } P, \not P \text{ isa } P \]

means

\[ \forall x: P(x) \rightarrow \neg P(x), \forall x: \neg P(x) \rightarrow P(x), \]

which is equivalent to

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- The set of formulas is not satisfiable, symbolically \( \Theta \models \).
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\begin{center}
P \isa \neg \ P, \ \neg \ P \isa \ P
\end{center}

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\begin{align*}
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\end{align*}

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The set of formulas is not satisfiable, symbolically $\Theta \models \bot$.

This is important to find out since in this case everything follows.
Theorem (Satisfiability of strict networks with negation)

\[ \Theta \models \text{iff the graph } G_{\Theta} \text{ contains a cycle from } \alpha \text{ to } \overline{\alpha} \text{ and back to } \alpha. \]

Proof.

\[ \iff \text{ Adding } \overline{\alpha_2} \rightarrow \overline{\alpha_1} \text{ corresponds to adding } \]

\[ \forall x: \neg \alpha_2(x) \rightarrow \neg \alpha_1(x) \]

when \( \forall x: \alpha_1(x) \rightarrow \alpha_2(x) \) is given. This is logically correct (contraposition). Since all directed paths in \( G_{\Theta} \) correspond to universally quantified implications that can be deduced from \( \pi(\Theta) \), a cycle as in the theorem implies:

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This, however, is unsatisfiable.
Deciding Satisfiability

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We have to show that unsatisfiability of $\Theta$ implies the existence of a cycle from $\alpha$ to $\overline{\alpha}$ and back to $\alpha$ in $G_\Theta$.

We prove the contraposition, i.e., that the absence of a cycle implies satisfiability.

We start with the universe $D = \{d\}$. Then we construct step-wise an interpretation for all concepts. Convention: When we assign $\alpha^I = \{d\}$, then we assign $\overline{\alpha}^I = \emptyset$ simultaneously.

1. Choose an $\alpha$ without an interpretation, which does not have a path to $\overline{\alpha}$.
2. Assign $\alpha^I = \{d\}$ and continue to do that for all concepts $\beta$ reachable from $\alpha$ which do not have an interpretation.
3. Continue until all concepts have an interpretation.

If there is still a concept without an interpretation, we always can find one satisfying the condition in step 1 since there is no cycle. In step 2, no concept above $\alpha$ can have an empty interpretation, so the assignment does not violate any subconcept relations.

$\implies$ When the assignment process finishes, we have a model!
Proof – continued.

⇒. We have to show that unsatisfiability of $\Theta$ implies the existence of a cycle from $\alpha$ to $\overline{\alpha}$ and back to $\alpha$ in $G_\Theta$.

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We prove the contraposition, i.e., that the absence of a cycle implies satisfiability.

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Proof – continued.

We have to show that unsatisfiability of $\Theta$ implies the existence of a cycle from $\alpha$ to $\overline{\alpha}$ and back to $\alpha$ in $G_\Theta$. We prove the contraposition, i.e., that the absence of a cycle implies satisfiability.

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⇝ When the assignment process finishes, we have a model!
isa Reasoning

Theorem (Inheritance in strict networks with negation)

\( \Theta \models \alpha_1 \text{isa} \alpha_2 \) iff one of the following conditions is satisfied:

1. \( \Theta \models \).
2. There is a path from \( \alpha_1 \) to \( \overline{\alpha_1} \) in \( G_\Theta \).
3. There is a path from \( \overline{\alpha_2} \) to \( \alpha_2 \) in \( G_\Theta \).
4. There is a path from \( \alpha_1 \) to \( \alpha_2 \) in \( G_\Theta \).

Proof sketch.

Soundness is obvious.
Completeness can be shown using the same argument that we used for completeness of the Satisfiability Theorem and the fact that we can start the construction process with \( \alpha_1^I = \{d\} \) and \( \overline{\alpha_2}^I = \{d\} \).

What about instance-relationship reasoning?
isa Reasoning

Theorem (Inheritance in strict networks with negation)

$$\Theta \models \alpha_1 \isa \alpha_2 \iff \text{one of the following conditions is satisfied:}$$

1. $$\Theta \models \top.$$
2. There is a path from $$\alpha_1$$ to $$\alpha'_1$$ in $$G_\Theta$$.
3. There is a path from $$\alpha'_2$$ to $$\alpha_2$$ in $$G_\Theta$$.
4. There is a path from $$\alpha_1$$ to $$\alpha_2$$ in $$G_\Theta$$.

Proof sketch.

Soundness is obvious. Completeness can be shown using the same argument that we used for completeness of the Satisfiability Theorem and the fact that we can start the construction process with $$\alpha_1^\mathcal{I} = \{d\}$$ and $$\alpha_2^\mathcal{I} = \{d\}$$.

What about instance-relationship reasoning?
isa Reasoning

Theorem (Inheritance in strict networks with negation)

\[ \Theta \models \alpha_1 \text{isa} \alpha_2 \text{ iff one of the following conditions is satisfied:} \]

1. \[ \Theta \models \alpha_1 \text{isa} \alpha_2 \]
2. \text{There is a path from } \alpha_1 \text{ to } \overline{\alpha_1} \text{ in } G_\Theta.
3. \text{There is a path from } \overline{\alpha_2} \text{ to } \alpha_2 \text{ in } G_\Theta.
4. \text{There is a path from } \alpha_1 \text{ to } \alpha_2 \text{ in } G_\Theta.

Proof sketch.

Soundness is obvious. Completeness can be shown using the same argument that we used for completeness of the Satisfiability Theorem and the fact that we can start the construction process with \( \alpha_1^\mathcal{T} = \{d\} \) and \( \overline{\alpha_2^\mathcal{T}} = \{d\} \).

What about instance-relationship reasoning?
Theorem (Inheritance in strict networks with negation)

\( \Theta \models \alpha_1 \text{ isa } \alpha_2 \) iff one of the following conditions is satisfied:

1. \( \Theta \models \neg \).
2. There is a path from \( \alpha_1 \) to \( \overline{\alpha_1} \) in \( G_\Theta \).
3. There is a path from \( \overline{\alpha_2} \) to \( \alpha_2 \) in \( G_\Theta \).
4. There is a path from \( \alpha_1 \) to \( \alpha_2 \) in \( G_\Theta \).

Proof sketch.

Soundness is obvious.
Completeness can be shown using the same argument that we used for completeness of the Satisfiability Theorem and the fact that we can start the construction process with \( \alpha_1^I = \{d\} \) and \( \overline{\alpha_2}^I = \{d\} \).

\( \therefore \) What about instance-relationship reasoning?
**Theorem (Inheritance in strict networks with negation)**

\[ \Theta \models \alpha_1 \text{ isa } \alpha_2 \text{ iff one of the following conditions is satisfied:} \]

1. \( \Theta \models \). 
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**Proof sketch.**

*Soundness* is obvious. *Completeness* can be shown using the same argument that we used for completeness of the Satisfiability Theorem and the fact that we can start the construction process with \( \alpha_1^\mathcal{I} = \{d\} \) and \( \overline{\alpha_2}^\mathcal{I} = \{d\} \).

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**Soundness** is obvious.

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What about instance-relationship reasoning?
Theorem (Inheritance in strict networks with negation)

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What about instance-relationship reasoning?
**Theorem (Inheritance in strict networks with negation)**

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1. \[ \Theta \models \neg \alpha_1 \]
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3. \text{There is a path from } \overline{\alpha_2} \text{ to } \alpha_2 \text{ in } G_\Theta. \]
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Soundness is obvious.
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What about instance-relationship reasoning?
A concept description is a concept name \((C)\), a negation of a concept name \((\text{not } C)\) or the conjunction of concept descriptions \((\alpha_1 \text{ and } \alpha_2)\).

**Example:**

\[(\text{Student and not Grad-Student}) \text{ isa } \text{Undergrad-Student}\]
\[(\text{Woman and Parent}) \text{ isa } \text{Mother}\]

- Logical semantics is obvious!
- Is it still possible to decide inheritance in polynomial time?
A concept description is a concept name ($C$), a negation of a concept name ($\text{not } C$) or the conjunction of concept descriptions ($\alpha_1 \text{ and } \alpha_2$).

**Example:**

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A Final Extension: Conjunctions and Negation

A **concept description** is a concept name \((C)\), a negation of a concept name \((\text{not } C)\) or the **conjunction** of concept descriptions \((\alpha_1 \text{ and } \alpha_2)\).

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**Example:**

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\begin{align*}
\text{(Student and not Grad-Student)} & \quad \text{isa} \quad \text{Undergrad-Student} \\
\text{(Woman and Parent)} & \quad \text{isa} \quad \text{Mother}
\end{align*}
\]

- **Logical semantics** is obvious!
- Is it still possible to decide inheritance in polynomial time?
Theorem (Complexity of strict inheritance with negation and conjunction)

The reasoning problem for strict inheritance networks with conjunction and negation is co-NP-hard.

Proof.

We show hardness by a reduction from 3SAT.
Let $D = C_1 \land \ldots \land C_n$ be formula in CNF with exactly three literals per clause (over atoms $a_i$). Let $\sigma(C_j)$ be the following translation:

- $a_1 \lor a_2 \lor a_3 \iff (\text{not } a_1 \text{ and not } a_2) \text{ isa } a_3$
- $\neg a_1 \lor a_2 \lor a_3 \iff (a_1 \text{ and not } a_2) \text{ isa } a_3$
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Extend $\sigma$ to CNF formulas.
Now it is easy to see that $D$ is unsatisfiable iff $\sigma(D) \models$. 
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Conclusion

- **Strict inheritance networks are easy**
  - Inheritance corresponds to a universally quantified implication
  - If concepts are atomic, everything can be decided in poly. time
  - We can deal with negation without increasing the complexity
  - Conjunction and negation, however, make the reasoning problem hard
  - ... as hard as propositional unsatisfiability.
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