Allen's Interval Calculus

Allen's Interval Calculus – Outline

Allen's Interval Calculus
- Motivation
- Intervals and Relations Between Them
- Processing an Example
- Composition Table
- Outlook

Reasoning in Allen's Interval Calculus

A Maximal Tractable Sub-Algebra

Literature

Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:
- NLP – we do not have precise time points
- Planning – we do not want to commit to time points too early
- Scenario descriptions – we do not have the exact times or do not want to state them

What are the primitives in our representation system?
- Time points: actions and events are instantaneous, or we consider their beginning and ending
- Time intervals: actions and events have duration
- Reducibility? Expressiveness? Computational costs for reasoning?
Motivation: Example
Consider a planning scenario for multimedia generation:

P1: Display Picture1
P2: Say “Put the plug in.”
P3: Say “The device should be shut off.”
P4: Point to Plug-in-Picture1.

Temporal relations between events:

P2 should happen during P1
P3 should happen during P1
P2 should happen before or directly precede P3
P4 should happen during or end together with P2

⇝ P4 happens before or directly precedes P3
⇝ We could add the statement “P4 does not overlap with P3” without creating an inconsistency.

Allen’s Interval Calculus

Allen’s interval calculus: time intervals and binary relations over them

Time intervals: \( X = (X^-, X^+) \), where \( X^- \) and \( X^+ \) are interpreted over the reals and \( X^- < X^+ \) (naïve approach)

Relations between concrete intervals, e.g.:

- \((1.0, 2.0)\) strictly before \((3.0, 5.5)\)
- \((1.0, 3.0)\) meets \((3.0, 5.5)\)
- \((1.0, 4.0)\) overlaps \((3.0, 5.5)\)

⇝ Which relations are conceivable?

The Base Relations

How many ways are there to order the four points of two intervals?

<table>
<thead>
<tr>
<th>Relation</th>
<th>Symbol</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>({X, Y) : (X^- &lt; X^+ &lt; Y^- &lt; Y^+)}</td>
<td>(&lt;)</td>
<td>before</td>
</tr>
<tr>
<td>({X, Y) : (X^- &lt; X^+ = Y^- &lt; Y^+)}</td>
<td>(m)</td>
<td>meets</td>
</tr>
<tr>
<td>({X, Y) : (X^- &lt; Y^- &lt; X^+ &lt; Y^+)}</td>
<td>(o)</td>
<td>overlaps</td>
</tr>
<tr>
<td>({X, Y) : (X^- = Y^- &lt; X^+ &lt; Y^+)}</td>
<td>(s)</td>
<td>starts</td>
</tr>
<tr>
<td>({X, Y) : (Y^- &lt; X^- &lt; X^+ = Y^+)}</td>
<td>(f)</td>
<td>finishes</td>
</tr>
<tr>
<td>({X, Y) : (Y^- &lt; X^- &lt; X^+ &lt; Y^+)}</td>
<td>(d)</td>
<td>during</td>
</tr>
<tr>
<td>({X, Y) : (Y^- = X^- &lt; X^+ = Y^+)}</td>
<td>(\equiv)</td>
<td>equal</td>
</tr>
</tbody>
</table>

and the converse relations (obtained by exchanging \(X\) and \(Y\))

⇝ These relations are JEPD.
Disjunctive Descriptions

- Assumption: We don’t have precise information about the relation between \( X \) and \( Y \), e.g.:

\[
X \circ Y \text{ or } X \mathbin{m} Y
\]

- \ldots modelled by sets of base relations (meaning the union of the relations):

\[
X \{o, m\} Y
\]

\( \sim 2^{13} \) imprecise relations (incl. \( \emptyset \) and \( B \))

Example of an indefinite qualitative description:

\[
\left\{ X \{o, m\} Y, Y \{m\} Z, X \{o, m\} Z \right\}
\]
Reasoning in Allen’s Interval Calculus

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Reasoning in Allen’s Interval Calculus

Constraint propagation algorithms (enforcing path consistency)

NP-Hardness Example

The Continuous Endpoint Class

Completeness for the CEP Class

A Maximal Tractable Sub-Algebra

Literature

Constraint Propagation – The Naive Algorithm

Enforcing path-consistency using the straight-forward method:

Let $Table[i, j]$ be an array of size $|n| \times |n|$ ($n$: number of intervals), in which we have recorded the constraints between the intervals.

**EnforcePathConsistency1** ($C$):

*Input:* a (binary) CSP $C = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP $C'$

**repeat**

*for* each pair $(i, j)$, $1 \leq i, j \leq n$

*for* each $k$ with $1 \leq k \leq n$

$Table[i, j] := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

**endfor**

**endfor**

**until** no entry in $Table$ is changed

⇝ terminates;

⇝ needs $O(n^5)$ intersections and compositions.

An $O(n^3)$ Algorithm

**EnforcePathConsistency2** ($C$):

*Input:* a (binary) CSP $C = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP $C'$

$Paths(i, j) = \{(i, j, k) : 1 \leq k \leq n\} \cup \{(k, i, j) : 1 \leq k \leq n\}$

$Queue := \bigcup_{i, j} Paths(i, j)$

**While** $Q \neq \emptyset$

*select* and *delete* $(i, k, j)$ from $Q$

$T := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

*if* $T \neq Table[i, j]$

$Table[i, j] := T$

$Table[j, i] := T^{-1}$

$Queue := Queue \cup Paths(i, j)$

*endif*

**endwhile**

Example for Incompleteness

A directed graph showing incompleteness in Allen’s Interval Calculus.
**NP-Hardness**

**Theorem (Kautz & Vilain)**

*CSAT is NP-hard for Allen’s interval calculus.*

**Proof.**

Reduction from 3-colorability (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability.

Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_i$</td>
<td>$m$, $m^{-1}$</td>
<td>2</td>
<td>$\forall v_i \in V$</td>
</tr>
</tbody>
</table>

$\forall (v_i, v_j) \in E$

This constraint system is satisfiable iff $G$ can be colored with 3 colors.

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**Looking for Special Cases**

- **Idea**: Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.

- **Note**: Interval formulae $X R Y$ can be expressed as clauses over atoms of the form $a op b$, where:
  - $a$ and $b$ are endpoints $X^-, X^+, Y^-$ and $Y^+$ and
  - $op \in \{<, >, =, \leq, \geq\}$.

- **Example**: All base relations can be expressed as unit clauses.

**Lemma**

Let $\pi(\Theta)$ be the translation of $\Theta$ to clause form. $\Theta$ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.

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**Why Do We Have Completeness?**

The set $\mathcal{C}$ is closed under intersection, composition, and converse (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

**Lemma**

*Each 3-consistent interval CSP over $\mathcal{C}$ is globally consistent.*

**Theorem (van Beek)**

Path consistency solves $\text{CMIN}(\mathcal{C})$ and decides $\text{CSAT}(\mathcal{C})$.

**Proof.**

Follows from the above lemma and the fact that a strongly $n$-consistent CSP is minimal.

**Corollary**

A path consistent interval CSP consisting of base relations only is satisfiable.
Helly’s Theorem

Definition
A set $M \subseteq \mathbb{R}^n$ is convex iff for all pairs of points $a, b \in M$, all points on the line connecting $a$ and $b$ belong to $M$.

Theorem (Helly)
Let $F$ be a family of at least $n + 1$ convex sets in $\mathbb{R}^n$. If all sub-families of $F$ with $n + 1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$.

Strong $n$-Consistency (1)

Proof.
We prove the claim by induction over $k$ with $k \leq n$.

Base case: $k = 1, 2, 3 \quad \checkmark$

Induction assumption: Assume strong $k - 1$-consistency (and non-emptiness of all relations)

Induction step: From the assumption, it follows that there is an instantiation of $k - 1$ variables $X_i$ to pairs $(s_i, e_i)$ satisfying the constraints $R_{ij}$ between the $k - 1$ variables.

We have to show that we can extend the instantiation to any $k$th variable.

Strong $n$-Consistency (2): Instantiating the $k$th Variable

Proof (Part 2).
The instantiation of the $k - 1$ variables $X_i$ to $(s_i, e_i)$ restricts the instantiation of $X_k$.

Note: Since $R_{ij} \in C$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \land e_j \geq X_k^- \land \ldots$$

Such inequalities define convex subsets in $\mathbb{R}^2$.

$\Rightarrow$ Consider sets of 3 inequalities (= 3 convex sets).

Strong $n$-Consistency (3): Using Helly’s Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only $X_k^-$ (or mention only $X_k^+$). Then it suffices to consider only 2 of these inequalities (the strongest). Because of $3$-consistency, there exists at least 1 common point satisfying these 3 inequalities.

Case 2: The inequalities mention $X_k^-$ and $X_k^+$, but it does not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. $X_k$) can be involved and by the same argument as above there exists a common point.

$\Rightarrow$ With Helly’s Theorem, it follows that there exists a consistent instantiation for all subsets of variables.

$\Rightarrow$ Strong $k$-consistency for all $k \leq n$. 
Outlook

- CMIN(\mathcal{C}) can be computed in \(O(n^3)\) time (for \(n\) being the number of intervals) using the path consistency algorithm.
- \(\mathcal{C}\) is a set of relations occurring “naturally” when observations are uncertain.
- \(\mathcal{C}\) contains 83 relations (incl. the impossible and the universal relations).
- Are there larger sets such that path consistency computes minimal CSPs? Probably not.
- Are there larger sets of relations that permit polynomial satisfiability testing? Yes.

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A Maximal Tractable Sub-Algebra

The EP-Subclass

**End-Point Subclass:** \(\mathcal{P} \subseteq \mathcal{A}\) is the subclass that permits a clause form containing only unit clauses (\(a \neq b\) is allowed).

**Example:** all basic relations and \(\{d, o\}\) since

\[
\pi(X \{d, o\} Y) = \{X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, X^+ < Y^+\}
\]

\[X \ldots X \ldots Y\]

**Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)**

*The path-consistency method decides CSAT(\(\mathcal{P}\)).*

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A Maximal Tractable Sub-Algebra

The ORD-Horn Subclass

**ORD-Horn Subclass:** \(\mathcal{H} \subseteq \mathcal{A}\) is the subclass that permits a clause form containing only Horn clauses, where only the following literals are allowed:

\[
a \leq b, a = b, a \neq b
\]

\(\neg a \leq b\) is not allowed!

**Example:** all \(R \in \mathcal{P}\) and \(\{o, s, f^{-1}\}\):

\[
\pi(X\{o, s, f^{-1}\}Y) = \{X^- \leq X^+, X^- \neq X^+, Y^- \leq Y^+, Y^- \neq Y^+, X^- \leq Y^-, X^- \leq Y^+, X^- \neq Y^+, X^+ \leq Y^+, Y^- \leq X^+, X^- \neq Y^-, X^+ \leq Y^+, X^- \neq Y^- \lor X^+ \neq Y^+\}
\]
Partial Orders: The ORD Theory

Let $ORD$ be the following theory:

- $\forall x, y, z: x \leq y \land y \leq z \rightarrow x \leq z$ (transitivity)
- $\forall x: x \leq x$ (reflexivity)
- $\forall x, y: x \leq y \land y \leq x \rightarrow x = y$ (anti-symmetry)
- $\forall x, y: x = y \rightarrow x \leq y$ (weakening of $=$)
- $\forall x, y: x = y \rightarrow y \leq x$ (weakening of $=$).

- $ORD$ describes partially ordered sets, $\leq$ being the ordering relation.
- $ORD$ is a Horn theory
- What is missing wrt to dense and linear orders?

Satisfiability over Partial Orders

Proposition

Let $\Theta$ be a CSP over $\mathcal{H}$. $\Theta$ is satisfiable over interval interpretations iff $\pi(\Theta) \cup ORD$ is satisfiable over arbitrary interpretations.

Proof.

$\Rightarrow$: Since the reals form a partially ordered set (i.e., satisfy $ORD$), this direction is trivial.

$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order.

Complexity of CSAT($\mathcal{H}$)

Let $ORD_{\pi(\Theta)}$ be the propositional theory resulting from instantiating all axioms with the endpoints occurring in $\pi(\Theta)$.

Proposition

$ORD \cup \pi(\Theta)$ is satisfiable iff $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is so.

Proof idea: Herbrand expansion!

Path-Consistency and the OH-Class

Lemma

Let $\Theta$ be a path-consistent set over $\mathcal{H}$. Then

$$(X\{\}Y) \notin \Theta \iff \Theta \text{ is satisfiable}$$

Proof Idea.

One can show that $ORD_{\pi(\Theta)} \cup \pi(\Theta)$ is closed wrt positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

Lemma

$\mathcal{H}$ is closed under intersection, composition, and conversion.

Theorem

The path-consistency method decides CSAT($\mathcal{H}$).

$\Rightarrow$ Maximal of $\mathcal{H}$?

$\Rightarrow$ Do we have to check all $8192 - 868$ extensions?
Complexity of Sub-Algebras

Let $\hat{\mathcal{S}}$ be the closure of $\mathcal{S} \subseteq \mathcal{A}$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by $\mathcal{S}$)

Theorem
$\text{CSAT}(\hat{\mathcal{S}})$ can be polynomially transformed to $\text{CSAT}(\mathcal{S})$.

Proof Idea.
All relations in $\hat{\mathcal{S}} - \mathcal{S}$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in $\mathcal{S}$, introducing perhaps some fresh variables.

$\implies$ Polynomiality of $\mathcal{S}$ extends to $\hat{\mathcal{S}}$.
$\implies$ NP-hardness of $\hat{\mathcal{S}}$ is inherited by all generating sets $\mathcal{S}$.

$\implies$ Note: $\mathcal{H} = \hat{\mathcal{H}}$.

“Interesting” Subclasses

Interesting subclasses of $\mathcal{A}$ should contain all basic relations. A computer-aided case analysis reveals: For $\mathcal{S} \supseteq \{\{B\} : B \in \mathcal{B}\}$ it holds that

1. $\hat{\mathcal{S}} \subseteq \mathcal{H}$, or
2. $N_1$ or $N_2$ is in $\hat{\mathcal{S}}$.

In case 2, one can show: $\text{CSAT}(\mathcal{S})$ is NP-complete.

$\implies$ $\mathcal{H}$ is the only maximal tractable subclass that is interesting.

Meanwhile, there is a complete classification of all sub-algebras containing at least one basic relation [IJCAI 2001] . . . but the question for sub-algebras not containing a basic relation is open.

Minimal Extensions of the $\mathcal{H}$-Subclass

A computer-aided case analysis leads to the following result:

Lemma
There are only two minimal sub-algebras that strictly contain $\mathcal{H}$: $\mathcal{X}_1$, $\mathcal{X}_2$

$N_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1$
$N_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2$

The clause form of these relations contain “proper” disjunctions!

Theorem
$\text{CSAT}(\mathcal{H} \cup \{N_i\})$ is NP-complete.

Question: Are there other maximal tractable subclasses?

Relevance?

Theoretical:
We now know the boundary between polynomial and NP-hard reasoning problems along the dimension expressiveness.

Practical:
All known applications either need only $\mathcal{P}$ or they need more than $\mathcal{H}$!

Backtracking methods might profit from the result because the branching factor is lower.

$\implies$ How difficult is $\text{CSAT}(\mathcal{A})$ in practice?

$\implies$ What are the relevant branching factors?
Solving General Allen CSPs

- Backtracking algorithm using path-consistency as a forward-checking method
- Relies on tractable fragments of Allen’s calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

Which tractable fragment should one use?

Branching Factors

- If the labels are split into base relations, then on average a label is split into 6.5 relations
- If the labels are split into pointizable relations ($P$), then on average a label is split into 2.955 relations
- If the labels are split into ORD-Horn relations ($\mathcal{H}$), then on average a label is split into 2.533 relations

A difference of 0.422

This makes a difference for “hard” instances.

Summary

- Allen’s interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the continuous endpoint class, minimal CSPs can be computed using the path-consistency method.
- For the larger ORD-Horn class, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.

Literature

A complete classification of complexity in Allen’s algebra in the presence of a non-trivial basic relation.