Allen’s Interval Calculus

Motivation
Intervals and Relations Between Them
Processing an Example
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Outlook

Reasoning in Allen’s Interval Calculus
Constraint propagation algorithms (enforcing path consistency)
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The Continuous Endpoint Class
Completeness for the CEP Class

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Allen’s Interval Calculus – Outline

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  Motivation
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Reasoning in Allen’s Interval Calculus

A Maximal Tractable Sub-Algebra

Literature
Qualitative Temporal Representation and Reasoning

Often we do not want to talk about precise times:

- **NLP** – we do not have precise time points
- **Planning** – we do not want to commit to time points too early
- **Scenario descriptions** – we do not have the exact times or do not want to state them

What are the primitives in our representation system?

- **Time points**: actions and events are instantaneous, or we consider their beginning and ending
- **Time intervals**: actions and events have duration
- **Reducibility? Expressiveness? Computational costs for reasoning?**
**Motivation: Example**

Consider a planning scenario for multimedia generation:

- **P1**: *Display Picture1*
- **P2**: *Say “Put the plug in.”*
- **P3**: *Say “The device should be shut off.”*
- **P4**: *Point to Plug-in-Picture1.*

Temporal relations between events:

- **P2** should happen during **P1**
- **P3** should happen during **P1**
- **P2** should happen before or directly precede **P3**
- **P4** should happen during or end together with **P2**

⇝ **P4** happens before or directly precedes **P3**

⇝ We could add the statement “**P4 does not overlap with P3**” without creating an inconsistency.
Allen’s Interval Calculus

- Allen’s interval calculus: time intervals and binary relations over them
- Time intervals: \( X = (X^-, X^+) \), where \( X^- \) and \( X^+ \) are interpreted over the reals and \( X^- < X^+ \) (naïve approach)
- Relations between concrete intervals, e.g.:
  
  \[
  (1.0, 2.0) \text{ strictly before } (3.0, 5.5) \\
  (1.0, 3.0) \text{ meets } (3.0, 5.5) \\
  (1.0, 4.0) \text{ overlaps } (3.0, 5.5) \\
  \ldots
  \]

Which relations are conceivable?
The Base Relations

How many ways are there to order the four points of two intervals?

<table>
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<th>Relation</th>
<th>Symbol</th>
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and the converse relations (obtained by exchanging $X$ and $Y$)

According to Nebel, Helmert, Wölfl (Uni Freiburg) KRR June 24, 2008 7 / 41

These relations are JEPD.
The 13 Base Relations Graphically
Disjunctive Descriptions

► Assumption: We don’t have precise information about the relation between $X$ and $Y$, e.g.:

$$X \circ Y \text{ or } X \mathbin{m} Y$$

► ... modelled by sets of base relations (meaning the union of the relations):

$$X \{o, m\} Y$$

$\Downarrow$ $2^{13}$ imprecise relations (incl. $\emptyset$ and $B$)

Example of an indefinite qualitative description:

$$\left\{ X \{o, m\} Y, \ Y \{m\} Z, \ X \{o, m\} Z \right\}$$
**Our Example . . . Formal**

P1: *Display Picture1*

P3: *Say “The device should be shut off.”*

P2: *Say “Put the plug in.”*

P4: *Point to Plug-in-Picture1.*

Compose the constraints: \( P4 \{d, f\} P2 \) and \( P2 \{d\} P1: \ P4 \{d\} P1. \)
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Using the **composition table** and the rules about operations on relations, we can **deduce** new relations between time intervals.

What would be a **systematic** approach?

How costly is that?

Is that **complete**?

If not, could it be complete on a subset of the relation system?
Reasoning in Allen’s Interval Calculus

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  Completeness for the CEP Class

A Maximal Tractable Sub-Algebra

Literature
Constraint Propagation – The Naive Algorithm

Enforcing path-consistency using the straight-forward method:
Let $Table[i, j]$ be an array of size $|n| \times |n|$ ($n$: number of intervals), in which we have recorded the constraints between the intervals.

**EnforcePathConsistency1** ($C$):

*Input:* a (binary) CSP $C = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP $C'$

repeat
  for each pair $(i, j)$, $1 \leq i, j \leq n$
    for each $k$ with $1 \leq k \leq n$
      $Table[i, j] := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$
    endfor
  endfor
until no entry in $Table$ is changed

$\Rightarrow$ terminates;

$\Rightarrow$ needs $O(n^5)$ intersections and compositions.
Reasoning in Allen’s Interval Calculus
Constraint propagation algorithms (enforcing path consistency)

An $O(n^3)$ Algorithm

**EnforcePathConsistency2 ($C$):**

*Input:* a (binary) CSP $C = \langle V, D, C \rangle$

*Output:* an equivalent, but path consistent CSP $C'$

$ Paths(i, j) = \{ (i, j, k) : 1 \leq k \leq n \} \cup \{ (k, i, j) : 1 \leq k \leq n \}$

$ Queue := \bigcup_{i,j} Paths(i, j)$

**While** $Q \neq \emptyset$

select and delete $(i, k, j)$ from $Q$

$ T := Table[i, j] \cap (Table[i, k] \circ Table[k, j])$

**if** $T \neq Table[i, j]$

$ Table[i, j] := T$

$ Table[j, i] := T^{-1}$

$ Queue := Queue \cup Paths(i, j)$

**endif**

endwhile
Example for Incompleteness

\[
\begin{align*}
A & \quad \{d, d^{-1}\} \quad \{f, f^{-1}\} \quad \{s, m\} \\
B & \quad \{f, f^{-1}\} \quad \{o\} \quad \{d, d^{-1}\} \\
C & \quad \{s, m\} \quad \{d, d^{-1}\} \\
D & \quad \{s, m\} \quad \{f, f^{-1}\}
\end{align*}
\]
NP-Hardness

Theorem (Kautz & Vilain)

CSAT is NP-hard for Allen’s interval calculus.

Proof.

Reduction from 3-colorability (original proof using 3Sat).

Let $G = (V, E)$, $V = \{v_1, \ldots, v_n\}$ be an instance of 3-colorability. Then we use the intervals $\{v_1, \ldots, v_n, 1, 2, 3\}$ with the following constraints:

\[
\begin{align*}
1 & \{m\} & 2 \\
2 & \{m\} & 3 \\
v_i & \{m, \equiv, m^{-1}\} & 2 & \forall v_i \in V \\
v_i & \{m, m^{-1}, \prec, \succ\} & v_j & \forall (v_i, v_j) \in E
\end{align*}
\]

This constraint system is satisfiable iff $G$ can be colored with 3 colors. 

Looking for Special Cases

- **Idea**: Let us look for polynomial special cases. In particular, let us look for sets of relations (subsets of the entire set of relations) that have an easy CSAT problem.

- **Note**: Interval formulae $X R Y$ can be expressed as clauses over atoms of the form $a \text{ op } b$, where:
  - $a$ and $b$ are endpoints $X^-, X^+, Y^-$ and $Y^+$ and
  - $\text{op} \in \{<, >, =, \leq, \geq\}$.

- **Example**: All base relations can be expressed as unit clauses.

**Lemma**

Let $\pi(\Theta)$ be the translation of $\Theta$ to clause form. $\Theta$ is satisfiable over intervals iff $\pi(\Theta)$ is satisfiable over the rational numbers.
The Continuous Endpoint Class

Continuous Endpoint Class $\mathcal{C}$: This is a subset of $\mathcal{A}$ such that there exists a clause form for each relation containing only unit clauses where $\neg(a = b)$ is forbidden.

Example: All basic relations and \{d, o, s\}, because

$$\pi(X \{d, o, s\} Y) = \{ X^- < X^+, Y^- < Y^+, X^- < Y^+, X^+ > Y^-, X^+ < Y^+ \}$$

\[\begin{array}{c}
\vdots \\
X \\
\vdots \\
\end{array}\quad \begin{array}{c}
\vdots \\
Y \\
\vdots \\
\end{array}\]
Why Do We Have Completeness?

The set $\mathcal{C}$ is closed under intersection, composition, and converse (it is a sub-algebra wrt. these three operations on relations). This can be shown by using a computer program.

**Lemma**

*Each 3-consistent interval CSP over $\mathcal{C}$ is globally consistent.*

**Theorem (van Beek)**

*Path consistency solves $\text{CMIN}(\mathcal{C})$ and decides $\text{CSAT}(\mathcal{C})$.*

**Proof.**

Follows from the above lemma and the fact that a strongly $n$-consistent CSP is minimal.

**Corollary**

*A path consistent interval CSP consisting of base relations only is satisfiable.*
Helly’s Theorem

Definition
A set $M \subseteq \mathbb{R}^n$ is convex iff for all pairs of points $a, b \in M$, all points on the line connecting $a$ and $b$ belong to $M$.

Theorem (Helly)
Let $F$ be a family of at least $n + 1$ convex sets in $\mathbb{R}^n$. If all sub-families of $F$ with $n + 1$ sets have a non-empty intersection, then $\bigcap F \neq \emptyset$. 
Strong $n$-Consistency (1)

Proof.
We prove the claim by induction over $k$ with $k \leq n$.

**Base case:** $k = 1, 2, 3$ \(\checkmark\)

**Induction assumption:** Assume strong $k - 1$-consistency (and non-emptiness of all relations)

**Induction step:** From the assumption, it follows that there is an instantiation of $k - 1$ variables $X_i$ to pairs $(s_i, e_i)$ satisfying the constraints $R_{ij}$ between the $k - 1$ variables.

We have to show that we can extend the instantiation to any $k$th variable.
Strong $n$-Consistency (2): Instantiating the $k$th Variable

Proof (Part 2).

The instantiation of the $k-1$ variables $X_i$ to $(s_i, e_i)$ restricts the instantiation of $X_k$.

**Note:** Since $R_{ij} \in C$ by assumption, these restrictions can be expressed by inequalities of the form:

$$s_i < X_k^+ \land e_j \geq X_k^- \land \ldots$$

Such inequalities define convex subsets in $\mathbb{R}^2$.

~~ Consider sets of 3 inequalities ($= 3$ convex sets).
Reasoning in Allen’s Interval Calculus Completeness for the CEP Class

Strong $n$-Consistency (3): Using Helly’s Theorem

Proof (Part 3).

Case 1: All 3 inequalities mention only $X_k^-$ (or mention only $X_k^+$). Then it suffices to consider only 2 of these inequalities (the strongest). Because of 3-consistency, there exists at least 1 common point satisfying these 3 inequalities.

Case 2: The inequalities mention $X_k^-$ and $X_k^+$, but it does not contain the inequality $X_k^- < X_k^+$. Then there are at most 2 inequalities with the same variable and we have the same situation as in Case 1.

Case 3: The set contains the inequality $X_k^- < X_k^+$. In this case, only three intervals (incl. $X_k$) can be involved and by the same argument as above there exists a common point.

$\Rightarrow$ With Helly’s Theorem, it follows that there exists a consistent instantiation for all subsets of variables.

$\Rightarrow$ Strong $k$-consistency for all $k \leq n$. 

Nebel, Helmert, Wölfl (Uni Freiburg)
Outlook

- CMIN(\(\mathcal{C}\)) can be computed in \(O(n^3)\) time (for \(n\) being the number of intervals) using the path consistency algorithm.
- \(\mathcal{C}\) is a set of relations occurring “naturally” when observations are uncertain.
- \(\mathcal{C}\) contains 83 relations (incl. the impossible and the universal relations).
- Are there larger sets such that path consistency computes minimal CSPs? Probably not
- Are there larger sets of relations that permit polynomial satisfiability testing? Yes
A Maximal Tractable Sub-Algebra

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Literature
The EP-Subclass

End-Point Subclass: \( \mathcal{P} \subseteq \mathcal{A} \) is the subclass that permits a clause form containing only unit clauses (\( a \neq b \) is allowed).

Example: all basic relations and \( \{d, o\} \) since

\[
\pi(X \{d, o\} Y) = \{ X^- < X^+, Y^- < Y^+, \\
X^- < Y^+, X^+ > Y^-, X^- \neq Y^-, \\
X^+ < Y^+ \} 
\]

\[ \prec \cdots X \cdots \prec \]

Theorem (Vilain & Kautz 86, Ladkin & Maddux 88)

The path-consistency method decides \( \text{CSAT}(\mathcal{P}) \).
The ORD-Horn Subclass

**ORD-Horn Subclass:** \( \mathcal{H} \subseteq \mathcal{A} \) is the subclass that permits a clause form containing only Horn clauses, where only the following literals are allowed:

\[
\begin{align*}
    a & \leq b, \quad a = b, \quad a \neq b \\
\neg a & \leq b \text{ is not allowed!}
\end{align*}
\]

**Example:** all \( R \in \mathcal{P} \) and \( \{o, s, f^{-1}\} \):

\[
\pi(X\{o, s, f^{-1}\}Y) = \begin{cases} 
    X^- \leq X^+, X^- \neq X^+ \\
    Y^- \leq Y^+, Y^- \neq Y^+ \\
    X^- \leq Y^- \\
    X^- \leq Y^+, X^- \neq Y^+ \\
    Y^- \leq X^+, X^+ \neq Y^- \\
    X^+ \leq Y^+ \\
    X^- \neq Y^- \lor X^+ \neq Y^+ 
\end{cases}
\]
Partial Orders: The \textit{ORD} Theory

Let \textit{ORD} be the following theory:

\begin{align*}
\forall x, y, z: & \quad x \leq y \land y \leq z \quad \rightarrow \quad x \leq z \quad \text{(transitivity)} \\
\forall x: & \quad x \leq x \quad \text{(reflexivity)} \\
\forall x, y: & \quad x \leq y \land y \leq x \quad \rightarrow \quad x = y \quad \text{(anti-symmetry)} \\
\forall x, y: & \quad x = y \quad \rightarrow \quad x \leq y \quad \text{(weakening of =)} \\
\forall x, y: & \quad x = y \quad \rightarrow \quad y \leq x \quad \text{(weakening of =)}.
\end{align*}

\begin{itemize}
\item \textit{ORD} describes partially ordered sets, \(\leq\) being the ordering relation.
\item \textit{ORD} is a Horn theory
\item What is missing wrt to dense and linear orders?
\end{itemize}
Satisfiability over Partial Orders

Proposition

Let $\Theta$ be a CSP over $\mathcal{H}$. $\Theta$ is satisfiable over interval interpretations iff $\pi(\Theta) \cup \text{ORD}$ is satisfiable over arbitrary interpretations.

Proof.

$\Rightarrow$: Since the reals form a partially ordered set (i.e., satisfy $\text{ORD}$), this direction is trivial.

$\Leftarrow$: Each extension of a partial order to a linear order satisfies all formulae of the form $a \leq b$, $a = b$, and $a \neq b$ which have been satisfied over the original partial order.
Complexity of CSAT(\(\mathcal{H}\))

Let \(ORD_{\pi(\Theta)}\) be the propositional theory resulting from instantiating all axioms with the endpoints occurring in \(\pi(\Theta)\).

Proposition

\(ORD \cup \pi(\Theta)\) is satisfiable iff \(ORD_{\pi(\Theta)} \cup \pi(\Theta)\) is so.

Proof idea: Herbrand expansion!

Theorem

CSAT(\(\mathcal{H}\)) can be decided in polynomial time.

Proof.

CSAT(\(\mathcal{H}\)) instances can be translated into a propositional Horn theory with blowup \(O(n^3)\) according to the previous Prop., and such a theory is decidable in polynomial time.
**Path-Consistency and the OH-Class**

**Lemma**

Let $\Theta$ be a path-consistent set over $\mathcal{H}$. Then

$$(X\{\} Y) \notin \Theta \text{ iff } \Theta \text{ is satisfiable}$$

**Proof Idea.**

One can show that $ORD_{\pi}(\Theta) \cup \pi(\Theta)$ is closed wrt positive unit resolution. Since this inference rule is refutation complete for Horn theories, the claim follows.

**Lemma**

$\mathcal{H}$ is closed under intersection, composition, and conversion.

**Theorem**

The path-consistency method decides $CSAT(\mathcal{H})$.

~~ Maximality of $\mathcal{H}$?

~~ Do we have to check all 8192 - 868 extensions?
Complexity of Sub-Algebras

Let $\hat{S}$ be the closure of $S \subseteq A$ under converse, intersection, and composition (i.e., the carrier of the least sub-algebra generated by $S$).

**Theorem**

$\text{CSAT}(\hat{S})$ can be polynomially transformed to $\text{CSAT}(S)$.

**Proof Idea.**

All relations in $\hat{S} - S$ can be modeled by a fixed number of compositions, intersections, and conversions of relations in $S$, introducing perhaps some fresh variables.

\[ \Rightarrow \text{Polynomiality of } S \text{ extends to } \hat{S}. \]

\[ \Rightarrow \text{NP-hardness of } \hat{S} \text{ is inherited by all generating sets } S. \]

\[ \Rightarrow \text{Note: } \mathcal{H} = \hat{\mathcal{H}}. \]
Minimal Extensions of the $\mathcal{H}$-Subclass

A computer-aided case analysis leads to the following result:

**Lemma**

There are only two minimal sub-algebras that strictly contain $\mathcal{H}$: $\mathcal{X}_1, \mathcal{X}_2$

\[
\mathcal{N}_1 = \{d, d^{-1}, o^{-1}, s^{-1}, f\} \in \mathcal{X}_1 \\
\mathcal{N}_2 = \{d^{-1}, o, o^{-1}, s^{-1}, f^{-1}\} \in \mathcal{X}_2
\]

The clause form of these relations contain “proper” disjunctions!

**Theorem**

$CSAT(\mathcal{H} \cup \{N_i\})$ is NP-complete.

**Question:** Are there other maximal tractable subclasses?
“Interesting” Subclasses

Interesting subclasses of $\mathcal{A}$ should contain all basic relations. A computer-aided case analysis reveals: For $S \supseteq \{\{B\} : B \in \mathcal{B}\}$ it holds that

1. $\hat{S} \subseteq \mathcal{H}$, or
2. $N_1$ or $N_2$ is in $\hat{S}$.

In case 2, one can show: $\text{CSAT}(S)$ is NP-complete.

$\implies \mathcal{H}$ is the only maximal tractable subclass that is interesting.

Meanwhile, there is a complete classification of all sub-algebras containing at least one basic relation [IJCAI 2001] . . . but the question for sub-algebras not containing a basic relation is open.
Relevance?

**Theoretical:**

We now know the boundary between polynomial and NP-hard reasoning problems along the dimension *expressiveness*.

**Practical:**

All known applications either need only $\mathcal{P}$ or they need more than $\mathcal{H}$!

Backtracking methods might profit from the result because the branching factor is lower.

How difficult is $\text{CSAT}(\mathcal{A})$ in practice?

What are the relevant branching factors?
Solving General Allen CSPs

- Backtracking algorithm using path-consistency as a forward-checking method
- Relies on tractable fragments of Allen’s calculus: split relations into relations of a tractable fragment, and backtrack over these.
- Refinements and evaluation of different heuristics

〜 Which tractable fragment should one use?
Branching Factors

- If the labels are split into base relations, then on average a label is split into

  6.5 relations

- If the labels are split into pointizable relations \((P)\), then on average a label is split into

  2.955 relations

- If the labels are split into ORD-Horn relations \((H)\), then on average a label is split into

  2.533 relations

\(\Rightarrow\) A difference of 0.422

\(\Rightarrow\) This makes a difference for “hard” instances.
Summary

- Allen’s interval calculus is often adequate for describing relative orders of events that have duration.
- The satisfiability problem for CSPs using the relations is NP-complete.
- For the continuous endpoint class, minimal CSPs can be computed using the path-consistency method.
- For the larger ORD-Horn class, CSAT is still decided by the path-consistency method.
- Can be used in practice for backtracking algorithms.
J. F. Allen.
Maintaining knowledge about temporal intervals.
Also in *Readings in Knowledge Representation*.

P. van Beek and R. Cohen.
Exact and approximate reasoning about temporal relations.

B. Nebel and H.-J. Bürckert.
Reasoning about temporal relations: A maximal tractable subclass of Allen’s interval algebra,

B. Nebel.
Solving hard qualitative temporal reasoning problems: Evaluating the efficiency of using the ORD-horn class.
A complete classification of complexity in Allen’s algebra in the presence of a non-trivial basic relation. 