Principles of Knowledge Representation and Reasoning

Nonmonotonic Reasoning II: Minimal Models and Nonmonotonic Logic Programs

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Conflicts between defaults in default logic lead to multiple extensions.

Each extension corresponds to a maximal set of non-violated defaults.

Reasoning with defaults can also be achieved by a simpler mechanism: predicate or propositional logic + minimize the number of cases where a default (expressed as a conventional formula) is violated.

⇒ minimal models

Notion of minimality: cardinality vs. set-inclusion.
Entailment with respect to Minimal Models

**Definition**

Let $A$ be a set of atomic propositions. Let $\Phi$ be a set of propositional formulae on $A$, and $B \subseteq A$ a set (called abnormalities).

Then $\Phi \models_B \psi$ ($\psi$ $B$-minimally follows from $\Phi$) if $\mathcal{I} \models \psi$ for all interpretations $\mathcal{I}$ such that $\mathcal{I} \models \Phi$ and there is no $\mathcal{I}'$ such that $\mathcal{I}' \models \Phi$ and $\{b \in B | \mathcal{I}' \models b\} \subsetneq \{b \in B | \mathcal{I} \models b\}$.
Minimal models: example

\[ \Phi = \{ \begin{align*}
\text{student} \land \neg \text{ABstudent} & \rightarrow \neg \text{earnsmoney}, & \text{student}, \\
\text{adult} \land \neg \text{ABadult} & \rightarrow \text{earnsmoney}, & \text{student} \rightarrow \text{adult}
\end{align*} \} \]

\( \Phi \) has the following models.

\( \mathcal{I}_1 \models \text{student} \land \text{adult} \land \text{earnsmoney} \land \text{ABstudent} \land \text{ABadult} \)

\( \mathcal{I}_2 \models \text{student} \land \text{adult} \land \neg \text{earnsmoney} \land \text{ABstudent} \land \text{ABadult} \)

\( \mathcal{I}_3 \models \text{student} \land \text{adult} \land \text{earnsmoney} \land \text{ABstudent} \land \neg \text{ABadult} \)

\( \mathcal{I}_4 \models \text{student} \land \text{adult} \land \neg \text{earnsmoney} \land \neg \text{ABstudent} \land \text{ABadult} \)
We can embed propositional minimal model reasoning in the propositional default logic.

**Theorem**

Let $A$ be a set of atomic propositions. Let $\Phi$ be a set of propositional formulae on $A$, and $B \subseteq A$. Then $\Phi \models_B \psi$ if and only if $\psi$ follows from $\langle D, W \rangle$ skeptically, where

$$D = \left\{ \frac{\neg b}{\neg b} \mid b \in B \right\} \text{ and } W = \Phi.$$
Proof sketch.

“⇒” : Assume there is extension $E$ of $\langle D, W \rangle$ such that $\psi \not\in E$. Hence there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models E$ and $\mathcal{I} \models \neg \psi$. By the fact that there is no extension $F$ such that $E \subset F$, $\mathcal{I}$ is a $B$-minimal model of $\Phi$. Hence $\psi$ does not $B$-minimally follow from $\Phi$.

“⇐” : Assume $\psi$ does not $B$-minimally follow from $\Phi$. Hence there is a $B$-minimal model $\mathcal{I}$ of $\Phi$ such that $\mathcal{I} \not\models \psi$. Define

$$E = \text{Th}(\Phi \cup \{ \neg b \mid b \in B, \mathcal{I} \models \neg b \}).$$

Now $\mathcal{I} \models E$ and because $\mathcal{I} \not\models \psi$, $\psi \not\in E$.

We can show that $E$ is an extension of $\langle D, W \rangle$.

Because there is an extension $E$ such that $\psi \not\in E$, $\psi$ does not skeptically follow from $\langle D, W \rangle$. 

\[
\square
\]
Relation to Default Logic: Proof

Proof sketch.

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Nonmonotonic Logic Programs: Background

- **Answer set semantics**: a formalization of negation-as-failure in logic programming (Prolog)
- Other formalizations: well-founded semantics, perfect-model semantics, inflationary semantics, ...
- Can be viewed as a simpler variant of default logic.
- A better alternative to the propositional logic in some applications.
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Nonmonotonic Logic Programs

- Rules $c \leftarrow b_1, \ldots, b_m, \text{not } d_1, \ldots, \text{not } d_k$
  where $\{c, b_1, \ldots, b_m, d_1, \ldots, d_k\} \subseteq A$ for a set $A = \{a_1, \ldots, a_n\}$ of propositions.
- Meaning similar to default logic: If
  1. we have derived $b_1, \ldots, b_m$ and
  2. cannot derive any of $d_1, \ldots, d_k$,
  then derive $c$.
- Rules without right-hand side: $c \leftarrow$
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Answer Sets – Formal Definition

- **Reduct** of a program $P$ with respect to a set of atoms $\Delta \subseteq A$:

$$P^\Delta := \{ c \leftarrow b_1, \ldots, b_m | (c \leftarrow b_1, \ldots, b_m, \text{not } d_1, \ldots, \text{not } d_k) \in P, \{d_1, \ldots, d_k\} \cap \Delta = \emptyset \}$$

- The closure $\text{dcl}(P) \subseteq A$ of a set $P$ of rules without not is defined by iterative application of the rules in the obvious way.

- A set of propositions $\Delta \subseteq A$ is an answer set of $P$ iff $\Delta = \text{dcl}(P^\Delta)$.
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Examples

- $P_1 = \{ a \leftarrow, \ b \leftarrow a, \ c \leftarrow b \}$
- $P_2 = \{ a \leftarrow b, \ b \leftarrow a \}$
- $P_3 = \{ p \leftarrow \text{not} \ p \}$
- $P_4 = \{ p \leftarrow \text{not} \ q, \ q \leftarrow \text{not} \ p \}$
- $P_5 = \{ p \leftarrow \text{not} \ q, \ q \leftarrow \text{not} \ p, \ \leftarrow p \}$
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Complexity: existence of answer sets is NP-complete

1. **Membership in NP:** Guess $\Delta \subseteq A$ (*nondet. polytime*), compute $P^\Delta$, compute its closure, compare to $\Delta$ (*everything det. polytime*).

2. **NP-hardness:** Reduction from 3SAT: an answer set exists iff clauses are satisfiable:

\[
\begin{align*}
  p & \leftarrow \neg \hat{p} \\
  \hat{p} & \leftarrow \neg p \\
  & \quad \text{for every proposition } p \text{ occurring in the clauses, and} \\
  & \quad \leftarrow \neg l'_1, \neg l'_2, \neg l'_3 \\
  & \quad \text{for every clause } l_1 \lor l_2 \lor l_3, \text{ where } l'_i = p \text{ if } l_i = p \text{ and } l'_i = \hat{p} \text{ if } l_i = \neg p.
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   \leftarrow \text{not} l_1', \text{not} l_2', \text{not} l_3'
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   for every clause $l_1 \lor l_2 \lor l_3$, where $l_i' = p$ if $l_i = p$ and $l_i' = \hat{p}$ if $l_i = \neg p$. 
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   for every clause $l_1 \lor l_2 \lor l_3$, where $l'_i = p$ if $l_i = p$ and $l'_i = \hat{p}$ if $l_i = \neg p$. 
Programs for Reasoning with Answer Sets

- smodels (Niemelä & Simons), dlv (Eiter et al.), ...
- Schematic input:

  ```prolog
  p(X) :- not q(X).  anc(X,Y) :- par(X,Y).
  q(X) :- not p(X).  anc(X,Y) :- par(X,Z), anc(Z,Y).
  r(a).              par(a,b). par(a,c). par(b,d).
  r(b).              female(a).
  r(c).              male(X) :- not(female(X)).
                    forefather(X,Y) :-
                                    anc(X,Y), male(X).  
  ```
Difference to the Propositional Logic

- The *ancestor* relation is the transitive closure of the *parent* relation.
- Transitive closure cannot be (concisely) represented in propositional/predicate logic.
  
  \[
  \text{par}(X,Y) \rightarrow \text{anc}(X,Y) \\
  \text{par}(X,Z) \land \text{anc}(Z,Y) \rightarrow \text{anc}(X,Y)
  \]

  The above formulae only guarantee that *anc* is a *superset* of the transitive closure of *par*.
- For transitive closure one needs the minimality condition in some form: nonmonotonic logics, fixpoint logics, ...
Stratification

The reason for multiple answer sets is the fact that \( a \) may depend on \( b \) and simultaneously \( b \) may depend on \( a \). The lack of this kind of circular dependencies makes reasoning easier.

**Definition**

A logic program \( P \) is **stratified** if \( P \) can be partitioned to \( P = P_1 \cup \cdots \cup P_n \) so that for all \( i \in \{1, \ldots, n\} \) and \((c \leftarrow b_1, \ldots, b_m, \text{not } d_1, \ldots, \text{not } d_k) \in P_i\),

1. there is no \textbf{not} \( c \) in \( P_i \) and
2. there are no occurrences of \( c \) anywhere in \( P_1 \cup \cdots \cup P_{i-1} \).
Theorem

A stratified program $P$ has exactly one answer set. The unique answer set can be computed in polynomial time.

Example

Our earlier examples with more than one or no answer sets:

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P_3 = \{ p \leftarrow \text{not} \, p \}\]
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P_4 = \{ p \leftarrow \text{not} \, q, \quad q \leftarrow \text{not} \, p \}\]
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Applications of Logic Programs

1. Simple forms of default reasoning (inheritance networks)

2. A solution to the frame problem: instead of using frame axioms, use defaults

\[ a_{t+1} \leftarrow a_t, \text{not } \neg a_{t+1} \]

By default, truth-values of facts stay the same.

3. Deductive databases (Datalog\(\neg\))

4. Et cetera: Everything that can be done with propositional logic can also be done with propositional nonmonotonic logic programs.
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Literature

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