Strategic Game

• A strategic game $G$ consists of
  – a finite set $N$ (the set of players)
  – for each player $i \in N$ a non-empty set $A_i$ (the set of actions or strategies available to player $i$), whereby $A = \bigcup_i A_i$
  – for each player $i \in N$ a function $u_i : A \to R$ (the utility or payoff function)
  – $G = (N, (A_i), (u_i))$

• If $A$ is finite, then we say that the game is finite
Playing the Game

- Each player $i$ makes a decision which action to play: $a_i$
- All players make their moves simultaneously leading to the action profile $a^* = (a_1, a_2, \ldots, a_n)$
- Then each player gets the payoff $u_i(a^*)$
- Of course, each player tries to maximize its own payoff, but what is the right decision?
- **Note:** While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!
  - If we want to model something like this, the payoff function must be changed
Notation

• For 2-player games, we use a matrix, where the strategies of player 1 are the rows and the strategies of player 2 the columns.

• The payoff for every action profile is specified as a pair \(x,y\), whereby \(x\) is the value for player 1 and \(y\) is the value for player 2.

• Example: For (T,R), player 1 gets \(x_{12}\), and player 2 gets \(y_{12}\).
Example Game:
Bach and Stravinsky

• Two people want to out together to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?

• This game is also called the *Battle of the Sexes*

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<tr>
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<th>Bach</th>
<th>Stravinsky</th>
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<tbody>
<tr>
<td>Bach</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Stravinsky</td>
<td>0,0</td>
<td>1,2</td>
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</table>
Example Game: Hawk-Dove

- Two animals fighting over some prey.
- Each can behave like a dove or a hawk.
- The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove.
- This game is also called *chicken*.

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<tr>
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<th>Dove</th>
<th>Hawk</th>
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<tr>
<td>Dove</td>
<td>3,3</td>
<td>1,4</td>
</tr>
<tr>
<td>Hawk</td>
<td>4,1</td>
<td>0,0</td>
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</table>
Example Game: Prisoner’s Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

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<td>Don’t confess</td>
<td>3,3</td>
<td>0,4</td>
</tr>
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<td>4,0</td>
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Solving a Game

• What is the right move?
• Different possible solution concepts
  – Elimination of strictly or weakly dominated strategies
  – Maximin strategies (for minimizing the loss in zero-sum games)
  – Nash equilibrium
• How difficult is it to compute a solution?
• Are there always solutions?
• Are the solutions unique?
Strictly Dominated Strategies

• Notation:
  – Let $a = (a_i)$ be a strategy profile
  – $a_{-i} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots a_n)$
  – $(a_{-i}, a'_i) := (a_1, \ldots, a_{i-1}, a'_i, a_{i+1}, \ldots a_n)$

• Strictly dominated strategy:
  – An strategy $a_j^* \in A_j$ is strictly dominated if there exists a strategy $a'_j$ such that for all strategy profiles $a \in A$:
    $$u_j(a_{-j}, a'_j) > u_j(a_{-j}, a_j^*)$$

• Of course, it is not rational to play strictly dominated strategies
Iterated Elimination of Strictly Dominated Strategies

• Since strictly dominated strategies will never be played, one can eliminate them from the game
• This can be done iteratively
• If this converges to a single strategy profile, the result is unique
• This can be regarded as the result of the game, because it is the only rational outcome
Iterated Elimination:  
Example

• Eliminate:
  , dominated by
  , dominated by
  , dominated by
  , dominated by

➢ Result:

<table>
<thead>
<tr>
<th></th>
<th>b1</th>
<th>b2</th>
<th>b3</th>
<th>b4</th>
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<tbody>
<tr>
<td>a1</td>
<td>1,7</td>
<td>2,5</td>
<td>7,2</td>
<td>0,1</td>
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<tr>
<td>a2</td>
<td>5,2</td>
<td>3,3</td>
<td>5,2</td>
<td>0,1</td>
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<tr>
<td>a3</td>
<td>7,0</td>
<td>2,5</td>
<td>0,4</td>
<td>0,1</td>
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<tr>
<td>a4</td>
<td>0,0</td>
<td>0,-2</td>
<td>0,0</td>
<td>9,-1</td>
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</table>
Iterated Elimination: Prisoner’s Dilemma

- Player 1 reasons that “not confessing” is strictly dominated and eliminates this option
- Player 2 reasons that player 1 will not consider “not confessing”. So he will eliminate this option for himself as well
- So, they both confess

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Weakly Dominated Strategies

• Instead of strict domination, we can also go for weak domination:
  – An strategy $a_{j^*} \in A_j$ is **weakly dominated** if there exists a strategy $a_j'$ such that for all strategy profiles $a \in A$:
    $$u_j(a_{-j}, a_j') \geq u_j(a_{-j}, a_{j^*})$$
    and for at least one profile $a \in A$:
    $$u_j(a_{-j}, a_j') > u_j(a_{-j}, a_{j^*}).$$
Results of Iterative Elimination of Weakly Dominated Strategies

- The result is not necessarily unique

- Example:
  - Eliminate
    - Eliminate:

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<tr>
<th></th>
<th>L</th>
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<tr>
<td><strong>T</strong></td>
<td>2,1</td>
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<td></td>
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<tr>
<td><strong>B</strong></td>
<td>0,0</td>
<td>1,1</td>
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Analysis of the **Guessing 2/3 of the Average Game**

- All strategies above 67 are weakly dominated, since they will *never ever* lead to winning the prize, so they can be eliminated!
- This means, that all strategies above \( \frac{2}{3} \times 67 \)

  can be eliminated
- … and so on
- … until all strategies above 1 have been eliminated!
- So: The rationale strategy would be to play 1!
Existence of Dominated Strategies

- Dominating strategies are a convincing solution concept
- Unfortunately, often dominated strategies do not exist
- What do we do in this case?
  - Nash equilibrium

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Nash Equilibrium

• A *Nash equilibrium* is an action profile $a^* \in A$ with the property that for all players $i \in N$:
  
  $$u_i(a^*) = u_i(a^*_{-i}, a^*_i) \geq u_i(a^*_{-i}, a_i) \quad \forall \; a_i \in A_i$$

• In words, it is an action profile such that there is no incentive for any agent to deviate from it

• While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept

• If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a *Nash equilibrium*
Example Nash-Equilibrium: Prisoner’s Dilemma

- Don’t – Don’t
  - not a NE
- Don’t – Confess (and vice versa)
  - not a NE
- Confess – Confess
  - NE

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</table>
Example Nash-Equilibrium: Hawk-Dove

- Dove-Dove:
  - not a NE
- Hawk-Hawk
  - not a NE
- Dove-Hawk
  - is a NE
- Hawk-Dove
  - is, of course, another NE

So, NEs are not necessarily unique
Auctions

- An object is to be assigned to a player in the set \{1,\ldots,n\} in exchange for a payment.
- Players' valuation of the object is \(v_i\), and \(v_1 > v_2 > \ldots > v_n\).
- The mechanism to assign the object is a sealed-bid auction: the players simultaneously submit bids (non-negative real numbers).
- The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment.
- The payment for a first price auction is the highest bid.
- What are the Nash equilibria in this case?
Formalization

• Game $G = (\{1, \ldots, n\}, (A_i), (u_i))$

• $A_i$: bids $b_i \in \mathbb{R}^+$

• $u_i(b_{-i}, b_i) = v_i - b_i$ if $i$ has won the auction, 0 otherwise

• Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.
Nash Equilibria for First-Price Sealed-Bid Auctions

• The Nash equilibria of this game are all profiles $b$ with:
  – $b_i \leq b_1$ for all $i \in \{2, \ldots, n\}$
    • No $i$ would bid more than $v_2$ because it could lead to negative utility
    • If a $b_i$ (with $< v_2$) is higher than $b_1$, player 1 could increase its utility by bidding $v_2 + \varepsilon$
    • So 1 wins in all NEs
  – $v_1 \geq b_1 \geq v_2$
    • Otherwise, player 1 either looses the bid (and could increase its utility by bidding more) or would have itself negative utility
  – $b_j = b_1$ for at least one $j \in \{2, \ldots, n\}$
    • Otherwise player 1 could have gotten the object for a lower bid
Another Game: Matching Pennies

- Each of two people chooses either **Head** or **Tail**. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.

- This is also a **zero-sum** or **strictly competitive** game.

- No NE at all! What shall we do here?

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<th>Tail</th>
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<tbody>
<tr>
<td><strong>Head</strong></td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
<td><strong>Tail</strong></td>
<td>-1,1</td>
<td>1,-1</td>
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Randomizing Actions …

• Since there does not seem to exist a rational decision, it might be best to randomize strategies.
• Play Head with probability $p$ and Tail with probability $1-p$.
• Switch to expected utilities.

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Some Notation

• Let $G = (N, (A_i), (u_i))$ be a strategic game

• Then $\Delta(A_i)$ shall be the set of probability distributions over $A_i$ – the set of mixed strategies $\alpha_i \in \Delta(A_i)$

• $\alpha_i(a_i)$ is the probability that $a_i$ will be chosen in the mixed strategy $\alpha_i$

• A profile $\alpha = (\alpha_i)$ of mixed strategies induces a probability distribution on $A$: $p(a) = \sum_i \alpha_i(a_i)$

• The expected utility is $U_i(\alpha) = \sum_{a \in A} p(a) u_i$
Example of a Mixed Strategy

- Let
  - $\alpha_1(H) = \frac{2}{3}, \alpha_1(T) = \frac{1}{3}$
  - $\alpha_2(H) = \frac{1}{3}, \alpha_2(T) = \frac{2}{3}$

- Then
  - $p(H,H) = \frac{2}{9}$
  - $p(H,T) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$
  - $p(T,H) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$
  - $p(T,T) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$
  - $U_1(\alpha_1, \alpha_2) = \frac{18}{26}$

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<tr>
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<td>-1,1</td>
</tr>
<tr>
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<td>-1,1</td>
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Mixed Extensions

• The **mixed extension** of the strategic game 
  \((N, (A_i), (u_i))\) is the strategic game \((N, \Delta(A_i), (U_i))\).

• The **mixed strategy Nash equilibrium** of a strategic game is a Nash equilibrium of its mixed extension.

• Note that the **Nash equilibria in pure strategies** (as studied in the last part) are just a special case of mixed strategy equilibria.
Nash’s Theorem

**Theorem.** Every finite strategic game has a mixed strategy Nash equilibrium.

- Note that it is essential that the game is **finite**
- So, there **exists** always a solution
- What is the **computational complexity**?
- Identifying a NE with a value larger than a particular value is **NP-hard**
The Support

• We call all pure actions $a_i$ that are chosen with non-zero probability by $\alpha_i$ the support of the mixed strategy $\alpha_i$.

**Lemma.** Given a finite strategic game, $\alpha^*$ is a mixed strategy equilibrium if and only if for every player $i$ every pure strategy in the support of $\alpha_i^*$ is a best response to $\alpha_{-i}^*$. 
Using the Support Lemma

- The **Support Lemma** can be used to compute all types of Nash equilibria in 2-person 2x2 action games.

  - There are 4 potential Nash equilibria in **pure strategies**
    - *Easy to check*
  
  - There are another 4 potential Nash equilibrium types with a **1-support** (pure) against **2-support** mixed strategies
    - Exists only if the corresponding pure strategy profiles are already Nash equilibria (follows from **Support Lemma**)

  - There exists one other potential Nash equilibrium type with a **2-support** against a **2-support** mixed strategies
    - Here we can use the **Support Lemma** to compute an NE (if there exists one)
A Mixed Nash Equilibrium for Matching Pennies

- There is clearly no NE in pure strategies
- Lets try whether there is a NE $\alpha^*$ in mixed strategies
- Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy $\alpha_2^*$

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<td>Tail</td>
<td>-1,1</td>
<td>1,-1</td>
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</tbody>
</table>

- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = U_1((0,1), (\alpha_2(H), \alpha_2(T)))$
- $U_1((1,0), (\alpha_2(H), \alpha_2(T))) = 1\alpha_2(H) + -1\alpha_2(T)$
- $U_1((0,1), (\alpha_2(H), \alpha_2(T))) = -1\alpha_2(H) + 1\alpha_2(T)$
- $\alpha_2(H) - \alpha_2(T) = -\alpha_2(H) + \alpha_2(T)$
- $2\alpha_2(H) = 2\alpha_2(T)$
- $\alpha_2(H) = \alpha_2(T)$
- Because of $\alpha_2(H) + \alpha_2(T) = 1$:
  - $\alpha_2(H) = \alpha_2(T) = 1/2$
  - Similarly for player 1!
- $U_1(\alpha^*) = 0$
Mixed NE for BoS

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<th>Bach</th>
<th>Stra-vinsky</th>
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<tr>
<td>Bach</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Stra-vinsky</td>
<td>0,0</td>
<td>1,2</td>
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- There are obviously 2 NEs in pure strategies.
- Is there also a strictly mixed NE?
- If so, again B and S played by player 1 should lead to the same payoff.

\[
\begin{align*}
U_1((1,0), (\alpha_2(B), \alpha_2(S))) &= U_1((0,1), (\alpha_2(B), \alpha_2(S))) \\
U_1((1,0), (\alpha_2(B), \alpha_2(S))) &= 2\alpha_2(B) + 0\alpha_2(S) \\
U_1((0,1), (\alpha_2(B), \alpha_2(S))) &= 0\alpha_2(B) + 1\alpha_2(S) \\
2\alpha_2(B) &= 1\alpha_2(S) \\
Because of \alpha_2(B) + \alpha_2(S) &= 1: \\
\alpha_2(B) &= 1/3 \\
\alpha_2(S) &= 2/3 \\
\text{Similarly for player 1!}
\end{align*}
\]

\[
U_1(\alpha^* ) = 2/3
\]
The 2/3 of Average Game

• You have $n$ players that are allowed to choose a number between 1 and $K$.
• The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners.
• What number would you play?
• What mixed strategy would you play?
A Nash Equilibrium in Pure Strategies

• All playing 1 is a NE in pure strategies
  – A deviation does not make sense

• All playing the same number different from 1 is not a NE
  – Choosing the number just below gives you more

• Similar, when all play different numbers, some not winning anything could get closer to 2/3 of the average and win something.

• So: Why did you not choose 1?

• Perhaps you acted rationally by assuming that the others do not act rationally?
Are there Proper Mixed Strategy Nash Equilibria?

• Assume there exists a mixed NE $\alpha$ different from the pure NE $(1,1,\ldots,1)$
  - Then there exists a maximal $k^* > 1$ which is played by some player with a probability $> 0$.
    – Assume player $i$ does so, i.e., $k^*$ is in the support of $\alpha_i$.
  - This implies $U_i(k^*,\alpha_{-i}) > 0$, since $k^*$ should be as good as all the other strategies of the support.
• Let $a$ be a realization of $\alpha$ s.t. $u_i(a) > 0$. Then at least one other player must play $k^*$, because not all others could play below $2/3$ of the average!
• In this situation player $i$ could get more by playing $k^*-1$.
  - This means, playing $k^*-1$ is better than playing $k^*$, i.e., $k^*$ cannot be in the support, i.e., $\alpha$ cannot be a NE
Summary

• **Strategic games** are one-shot games, where everybody plays its move simultaneously.
• Each player gets a payoff based on its **payoff function** and the resulting **action profile**.
• **Iterated elimination of strictly dominated strategies** is a convincing solution concept.
• **Nash equilibrium** is another solution concept: Action profiles, where **no player has an incentive to deviate**.
• It also might **not be unique** and there can be even infinitely many NEs or none at all!

> For every finite strategic game, there exists a Nash equilibrium in **mixed strategies**.
• Actions in the support of mixed strategies in a NE are always best answers to the NE profile, and therefore have the same payoff \( \sim \) **Support Lemma**.
• Computing a NE in mixed strategies is NP-hard.