A strategic game consists of:
- A finite set \( N \) (the set of players)
- For each player \( i \in N \) a non-empty set \( A_i \) (the set of actions or strategies available to player \( i \)), whereby \( A = \bigcup_{i \in N} A_i \)
- For each player \( i \in N \) a function \( u_i : A \rightarrow R \) (the utility or payoff function)
- \( G = (N, (A_i), (u_i)) \)

If \( A \) is finite, then we say that the game is finite.

Each player \( i \) makes a decision which action to play: \( a_i \).

All players make their moves simultaneously leading to the action profile \( a^* = (a_1, a_2, \ldots, a_n) \).

Then each player gets the payoff \( u_i(a^*) \).

Of course, each player tries to maximize its own payoff, but what is the right decision?

Note: While we want to maximize our payoff, we are not interested in harming our opponent. It just does not matter to us what he will get!

- If we want to model something like this, the payoff function must be changed.
Example Game: Bach and Stravinsky

- Two people want to go to a concert of music by either Bach or Stravinsky. Their main concern is to go out together, but one prefers Bach, the other Stravinsky. Will they meet?
- This game is also called the Battle of the Sexes.

<table>
<thead>
<tr>
<th></th>
<th>Bach</th>
<th>Stravinsky</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bach</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Stravinsky</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

Example Game: Hawk-Dove

- Two animals fighting over some prey.
- Each can behave like a dove or a hawk.
- The best outcome is if oneself behaves like a hawk and the opponent behaves like a dove.
- This game is also called chicken.

<table>
<thead>
<tr>
<th></th>
<th>Dove</th>
<th>Hawk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dove</td>
<td>3,3</td>
<td>1,4</td>
</tr>
<tr>
<td>Hawk</td>
<td>4,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Example Game: Prisoner’s Dilemma

- Two suspects in a crime are put into separate cells.
- If they both confess, each will be sentenced to 3 years in prison.
- If only one confesses, he will be freed.
- If neither confesses, they will both be convicted of a minor offense and will spend one year in prison.

<table>
<thead>
<tr>
<th></th>
<th>Don’t confess</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Don’t confess</td>
<td>3,3</td>
<td>0,4</td>
</tr>
<tr>
<td>Confess</td>
<td>4,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Solving a Game

- What is the right move?
- Different possible solution concepts
  - Elimination of strictly or weakly dominated strategies
  - Maximin strategies (for minimizing the loss in zero-sum games)
  - Nash equilibrium
- How difficult is it to compute a solution?
- Are there always solutions?
- Are the solutions unique?
Strictly Dominated Strategies

• Notation:
  – Let \( a = (a_i) \) be a strategy profile
  – \( a_i := (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n) \)
  – \( (a_i, a') := (a_1, ..., a_{i-1}, a', a_{i+1}, ..., a_n) \)

• Strictly dominated strategy:
  – An strategy \( a_j^* \in A_j \) is strictly dominated if there exists a strategy \( a'_j \) such that for all strategy profiles \( a \in A \):
    \[ u_j(a_j, a'_j) > u_j(a_j, a_j^*) \]

• Of course, it is not rational to play strictly dominated strategies

Iterated Elimination of Strictly Dominated Strategies

• Since strictly dominated strategies will never be played, one can eliminate them from the game
• This can be done iteratively
• If this converges to a single strategy profile, the result is unique
• This can be regarded as the result of the game, because it is the only rational outcome

Iterated Elimination:

Example

• Eliminate:
  , dominated by
  , dominated by
  , dominated by
  , dominated by
  , dominated by

\[ \begin{array}{cccc}
  & b1 & b2 & b3 & b4 \\
  a1 & 1,7 & 2,5 & 7,2 & 0,1 \\
a2 & 5,2 & 3,3 & 5,2 & 0,1 \\
a3 & 7,0 & 2,5 & 0,4 & 0,1 \\
a4 & 0,0 & 0,-2 & 0,0 & 9,-1 \\
\end{array} \]

➢ Result:

Iterated Elimination: Prisoner’s Dilemma

• Player 1 reasons that “not confessing” is strictly dominated and eliminates this option
• Player 2 reasons that player 1 will not consider “not confessing”. So he will eliminate this option for himself as well
• So, they both confess

<table>
<thead>
<tr>
<th>Don’t confess</th>
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</tr>
</thead>
<tbody>
<tr>
<td>3,3</td>
<td>0,4</td>
</tr>
<tr>
<td>4,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>
Weakly Dominated Strategies

- Instead of strict domination, we can also go for weak domination:
  - An strategy \( a_j^* \in A_j \) is weakly dominated if there exists a strategy \( a_j' \) such that for all strategy profiles \( a \in A \):
    \[
    u_j(a_j, a_j') \geq u_j(a_j, a_j^*)
    \]
    and for at least one profile \( a \in A \):
    \[
    u_j(a_j, a_j') > u_j(a_j, a_j^*).
    \]

Results of Iterative Elimination of Weakly Dominated Strategies

- The result is not necessarily unique
- Example:
  - Eliminate
    \[
    \begin{array}{c}
    T \quad 2,1 \quad 0,0 \\
    M \quad 2,1 \quad 1,1 \\
    B \quad 0,0 \quad 1,1
    \end{array}
    \]
  - Eliminate:

Analysis of the Guessing 2/3 of the Average Game

- All strategies above 67 are weakly dominated, since they will never ever lead to winning the prize, so they can be eliminated!
- This means, that all strategies above \( 2/3 \times 67 \)
  can be eliminated
- … and so on
- … until all strategies above 1 have been eliminated!
- So: The rationale strategy would be to play 1!

Existence of Dominated Strategies

- Dominating strategies are a convincing solution concept
- Unfortunately, often dominated strategies do not exist
- What do we do in this case?
  - Nash equilibrium
Nash Equilibrium

• A Nash equilibrium is an action profile \( a^* \in A \) with the property that for all players \( i \in N \):
  \[
  u_i(a^*) = u_i(a^*, a^*) \geq u_i(a^*, a) \forall a \in A_i
  \]
• In words, it is an action profile such that there is no incentive for any agent to deviate from it
• While it is less convincing than an action profile resulting from iterative elimination of dominated strategies, it is still a reasonable solution concept
• If there exists a unique solution from iterated elimination of strictly dominated strategies, then it is also a Nash equilibrium

Example Nash-Equilibrium: Prisoner’s Dilemma

<table>
<thead>
<tr>
<th></th>
<th>Don’t confess</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Don’t – Don’t</td>
<td>3,3</td>
<td>0,4</td>
</tr>
<tr>
<td>Don’t – Confess (and vice versa)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Confess – Confess</td>
<td>4,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Example Nash-Equilibrium: Hawk-Dove

<table>
<thead>
<tr>
<th></th>
<th>Dove</th>
<th>Hawk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dove-Dove:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>– not a NE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hawk-Hawk:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>– not a NE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dove-Hawk:</td>
<td>Dove</td>
<td></td>
</tr>
<tr>
<td>– is a NE</td>
<td>3,3</td>
<td>1,4</td>
</tr>
<tr>
<td>Hawk-Dove:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>– is, of course, another NE</td>
<td>Dove</td>
<td></td>
</tr>
<tr>
<td>So, NEs are not necessarily unique</td>
<td>4,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Auctions

• An object is to be assigned to a player in the set \{1, \ldots, n\} in exchange for a payment.
• Players' valuation of the object is \( v_i \), and \( v_1 > v_2 > \ldots > v_n \).
• The mechanism to assign the object is a sealed-bid auction: the players simultaneously submit bids (non-negative real numbers)
• The object is given to the player with the lowest index among those who submit the highest bid in exchange for the payment
• The payment for a first price auction is the highest bid.
• What are the Nash equilibria in this case?
Formalization

• Game \( G = (\{1, \ldots, n\}, (A_i), (u_i)) \)
• \( A_i \): bids \( b_i \in \mathbb{R}^+ \)
• \( u_i(b_{-i}, b_i) = v_i - b_i \) if \( i \) has won the auction, 0 otherwise
• Nobody would bid more than his valuation, because this could lead to negative utility, and we could easily achieve 0 by bidding 0.

Nash Equilibria for First-Price Sealed-Bid Auctions

• The Nash equilibria of this game are all profiles \( b \) with:
  - \( b_i \leq b_1 \) for all \( i \in \{2, \ldots, n\} \)
    - No \( i \) would bid more than \( v_2 \) because it could lead to negative utility
    - If a \( b_i \) (with \( < v_2 \)) is higher than \( b \), player 1 could increase its utility by bidding \( v_2 + \epsilon \)
    - So 1 wins in all NEs
  - \( v_i \geq b_1 \geq v_2 \)
    - Otherwise, player 1 either loses the bid (and could increase its utility by bidding more) or would have itself negative utility
  - \( b_j = b_1 \) for at least one \( j \in \{2, \ldots, n\} \)
    - Otherwise player 1 could have gotten the object for a lower bid

Another Game: Matching Pennies

• Each of two people chooses either Head or Tail. If the choices differ, player 1 pays player 2 a euro; if they are the same, player 2 pays player 1 a euro.
• This is also a zero-sum or strictly competitive game
• No NE at all! What shall we do here?

Randomizing Actions …

• Since there does not seem to exist a rational decision, it might be best to randomize strategies.
• Play Head with probability \( p \) and Tail with probability \( 1-p \)
• Switch to expected utilities
Some Notation

- Let $G = (N, (A_i), (u_i))$ be a strategic game.
- Then $\Delta(A_i)$ shall be the set of probability distributions over $A_i$—the set of mixed strategies $\alpha_i \in \Delta(A_i)$.
- $\alpha_i(a_i)$ is the probability that $a_i$ will be chosen in the mixed strategy $\alpha_i$.
- A profile $\alpha = (\alpha_i)$ of mixed strategies induces a probability distribution on $A$: $p(a) = \sum_{\alpha_i \in \Delta(A_i)} \alpha_i(a_i)$.
- The expected utility is $U_i(\alpha) = \sum_{a \in A} p(a) u_i(a)$.

Example of a Mixed Strategy

- Let $\alpha_1(H) = 2/3$, $\alpha_1(T) = 1/3$.
- $\alpha_2(H) = 1/3$, $\alpha_2(T) = 2/3$.
- Then $p(H,H) = 2/9$, $p(H,T) = 1/9$, $p(T,H) = 1/9$, $p(T,T) = 2/9$.
- $U_1(\alpha_1, \alpha_2) = 18/25$.

<table>
<thead>
<tr>
<th></th>
<th>Head</th>
<th>Tail</th>
</tr>
</thead>
<tbody>
<tr>
<td>Head</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
<td>Tail</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
</tbody>
</table>

Mixed Extensions

- The mixed extension of the strategic game $(N, (A_i), (u_i))$ is the strategic game $(N, \Delta(A_i), (U_i))$.
- The mixed strategy Nash equilibrium of a strategic game is a Nash equilibrium of its mixed extension.
- Note that the Nash equilibria in pure strategies (as studied in the last part) are just a special case of mixed strategy equilibria.

Nash’s Theorem

**Theorem.** Every finite strategic game has a mixed strategy Nash equilibrium.

- Note that it is essential that the game is finite.
- So, there exists always a solution.
- What is the computational complexity?
- Identifying a NE with a value larger than a particular value is NP-hard.
A Mixed Nash Equilibrium for Matching Pennies

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<td>-1,1</td>
<td>1,-1</td>
</tr>
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- There is clearly no NE in pure strategies.
- Let's try whether there is a NE $\alpha^*$ in mixed strategies.
- Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy $\alpha_1^*$.

$U_1((1,0), (\alpha_2(H), \alpha_2(T))) = U_1((0,1), (\alpha_2(H), \alpha_2(T)))$

$U_1((1,0), (\alpha_2(H), \alpha_2(T))) = 1\alpha_2(H) + -1\alpha_2(T)$

$U_1((0,1), (\alpha_2(H), \alpha_2(T))) = -1\alpha_2(H) + 1\alpha_2(T)$

$\alpha_2(H) - \alpha_2(T) = \alpha_2(H) + \alpha_2(T)$

$2\alpha_2(H) = 2\alpha_2(T)$

$\alpha_2(H) = \alpha_2(T)$

Because of $\alpha_2(H) + \alpha_2(T) = 1$:

$\alpha_2(H) = \alpha_2(T) = 1/2$

Similarly for player 1!

$U_1(\alpha^*) = 0$

The Support

- We call all pure actions $a_i$ that are chosen with non-zero probability by $\alpha_i$ the support of the mixed strategy $\alpha_i$.

**Lemma.** Given a finite strategic game, $\alpha^*$ is a mixed strategy equilibrium if and only if for every player $i$ every pure strategy in the support of $\alpha_i^*$ is a best response to $\alpha_{-i}^*$.

Mixed NE for BoS

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</tr>
<tr>
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<td>1,2</td>
</tr>
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</table>

- There are obviously 2 NEs in pure strategies.
- Let's try whether there is a NE $\alpha^*$ in mixed strategies.
- Then the H action by player 1 should have the same utility as the T action when played against the mixed strategy $\alpha_1^*$.

$U_1((1,0), (\alpha_2(B), \alpha_2(S))) = U_1((0,1), (\alpha_2(B), \alpha_2(S)))$

$U_1((1,0), (\alpha_2(B), \alpha_2(S))) = 2\alpha_2(B) + 0\alpha_2(S)$

$U_1((0,1), (\alpha_2(B), \alpha_2(S))) = 0\alpha_2(B) + 1\alpha_2(S)$

$2\alpha_2(B) = 1\alpha_2(S)$

Because of $\alpha_2(B) + \alpha_2(S) = 1$:

$\alpha_2(B) = 1/3$

$\alpha_2(S) = 2/3$

Similarly for player 1!

$U_1(\alpha^*) = 2/3$
The 2/3 of Average Game

• You have \( n \) players that are allowed to choose a number between 1 and \( K \).
• The players coming closest to 2/3 of the average over all numbers win. A fixed prize is split equally between all the winners.
• What number would \( \text{you} \) play?
• What \text{mixed strategy} would you play?

A Nash Equilibrium in Pure Strategies

• All playing 1 is a NE in pure strategies
  – A deviation does not make sense
• All playing the same number different from 1 is not a NE
  – Choosing the number just below gives you more
• Similar, when all play different numbers, some not winning anything could get closer to 2/3 of the average and win something.
• So: \textbf{Why did you not choose 1?}
• Perhaps \text{you acted rationally} by assuming that the others do not act rationally?

Are there Proper Mixed Strategy Nash Equilibria?

• Assume there exists a mixed NE \( \alpha \) different from the pure NE \( (1,1,...,1) \)
• Then there exists a maximal \( k^* > 1 \) which is played by some player with a probability \( > 0 \).
  – Assume player \( i \) does so, i.e., \( k^* \) is in the support of \( \alpha_i \).
• This implies \( U_i(k^*,\alpha_{-i}) > 0 \), since \( k^* \) should be as good as all the other strategies of the support.
• Let \( a \) be a realization of \( \alpha \) s.t. \( u(a) > 0 \). Then at least one other player must play \( k^* \), because not all others could play below 2/3 of the average!
• In this situation player \( i \) could get more by playing \( k^*-1 \).
• This means, playing \( k^*-1 \) is better than playing \( k^* \), i.e., \( k^* \) cannot be in the support, i.e., \( \alpha \) cannot be a NE

Summary

• Strategic games are one-shot games, where everybody plays its move simultaneously.
• Each player gets a payoff based on its payoff function and the resulting action profile.
• Iterated elimination of strictly dominated strategies is a convincing solution concept.
• Nash equilibrium is another solution concept: Action profiles, where no player has an incentive to deviate.
• It also might \text{not be unique} and there can be even infinitely many NEs or none at all!
  ➢ For every finite strategic game, there exists a Nash equilibrium in \text{mixed strategies}.
• Actions in the support of mixed strategies in a NE are always best answers to the NE profile, and therefore have the same payoff \( \sim \) \text{Support Lemma}.
• Computing a NE in mixed strategies is NP-hard.