Motivation

- Usually:
  - **Given:** A logical theory (set of propositions)
  - **Question:** Does a proposition logically follow from this theory?
  - Reduction to unsatisfiability, which is coNP-complete (complementary to NP problems)
- Sometimes:
  - **Given:** A logical theory
  - **Wanted:** Model of the theory.
  - **Example:** Configurations that fulfill the constraints given in the theory.
  - Can be “easier” because it is enough to find one model

The Davis-Putnam Procedure

**DP Function**

Given a set of clauses $\Delta$ defined over a set of variables $\Sigma$, return "satisfiable" if $\Delta$ is satisfiable. Otherwise return "unsatisfiable".

1. If $\Delta = \emptyset$ return "satisfiable"
2. If $\Box \in \Delta$ return "unsatisfiable"
3. **Unit-propagation Rule:** If $\Delta$ contains a unit-clause $C$, assign a truth-value to the variable in $C$ that satisfies $C$, simplify $\Delta$ to $\Delta'$ and return $DP(\Delta')$.
4. **Splitting Rule:** Select from $\Sigma$ a variable $v$ which has not been assigned a truth-value. Assign one truth value $t$ to it, simplify $\Delta$ to $\Delta'$ and call $DP(\Delta')$.
   a. If the call returns "satisfiable", then return "satisfiable"
   b. Otherwise assign the other truth-value to $v$ in $\Delta$, simplify to $\Delta''$ and return $DP(\Delta'')$. 
Properties of DP

- DP is complete, correct, and guaranteed to terminate.
- DP constructs a model, if one exists.
- In general, DP requires exponential time (splitting rule!)
- DP is polynomial on horn clauses, i.e., clauses with at most one positive literal. \((\neg A_1 \lor \ldots \lor \neg A_n \lor B \equiv \land_i A_i \rightarrow B)\)
- **Heuristics** are needed to determine which variable should be instantiated next and which value should be used.
- In all SAT competitions so far, DP-based procedures have shown the best performance.

DP on Horn Clauses (1)

Note:

1. The simplifications in DP on Horn clauses always generate **Horn clauses**.
2. A set of Horn clauses without unit clauses is satisfiable
   - *All clauses have at least one negative literal*
   - *Assign false to all variables*
3. If the first sequence of applications of the unit propagation rule in DP does not lead to the empty clause, a set of Horn clauses without unit clauses is generated (which is satisfiable according to (2))
**DP on Horn Clauses (2)**

4. Although a set of Horn clauses without a unit clause is satisfiable, DP may **not immediately recognize** it.
   a. If DP assigns *false* to a variable, this cannot lead to an unsatisfiable set and after a sequence of unit propagations we are in **the same situation** as in 4.
   b. If DP assigns *true*, then we may get an empty clause - perhaps after unit propagation (and have to backtrack) - or the set is still satisfiable and we are in **the same situation** as in 4.

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**DP on Horn Clauses (3)**

In summary:

1. DP executes a **sequence of unit propagation steps** resulting in
   - an empty clause or
   - a set of Horn clauses without a unit clause, which is satisfiable

2. In the latter case, DP proceeds by **choosing** for one variable:
   - *false*, which does not change the satisfiability
   - *true*, which either
     - leads to an immediate contradiction (after unit propagation) and backtracking or
     - does not change satisfiability

> Run time is *polynomial* in the number of variables

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**How Good is DP in the Average Case?**

- We know that SAT is NP-complete, i.e., in the worst case, it takes exponential time.
- This is clearly also true for the DP-procedure.
  → Couldn’t we do better in the **average case**?
- For CNF-formulae in which the probability for a positive appearance, negative appearance and non-appearance in a clause is 1/3, DP needs on average **quadratic time** (Goldberg 79)!
  → The probability that these formulae are satisfiable is, however, very high.

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**Phase Transitions ...**

Conversely, we can, of course, try to identify hard to solve problem instances.

Cheeseman et al. (IJCAI-91) came up with the following plausible conjecture:

> All NP-complete problems have at least **one order parameter** and the hard to solve problems are around a critical value of this order parameter. This critical value (a **phase transition**) separates one region from another, such as over-constrained and under-constrained regions of the problem space.

Confirmation for graph coloring and Hamilton path ... later also for other NP-complete problems.
Phase Transitions with 3-SAT

Constant clause length model (Mitchell et al., AAAI-92):
Clause length \( k \) is given. Choose variables for every clause \( k \) and use the complement with probability 0.5 for each variable.

Phase transition for 3-SAT with a clause/variable ratio of approx. 4.3:

Notes on the Phase Transition

- When the probability of a solution is close to 1 (under-constrained), there are many solutions, and the first search path of a backtracking search is usually successful.
- If the probability of a solution is close to 0 (over-constrained), this fact can usually be determined early in the search.
- In the phase transition stage, there are many near successes (“close, but no cigar”).
  → (limited) possibility of predicting the difficulty of finding a solution based on the parameters.
  → (search intensive) benchmark problems are located in the phase region (but they have a special structure)

Empirical Difficulty

The Davis-Putnam (DP) Procedure shows extreme runtime peaks at the phase transition:

Local Search Methods for Solving Logical Problems

In many cases, we are interested in finding a satisfying assignment of variables (example CSP), and we can sacrifice completeness if we can "solve" much large instances this way.

Standard process for optimization problems: Local Search

- Based on a (random) configuration
- Through local modifications, we hope to produce better configurations
  → Main problem: local maxima
Dealing with Local Maxima

As a measure of the value of a configuration in a logical problem, we could use the number of satisfied constraints/clauses.

But local search seems inappropriate, considering we want to find a global maximum (all constraints/clauses satisfied).

By restarting and/or injecting noise, we can often escape local maxima.

Actually: Local search performs very well for finding satisfying assignments of CNF formulae (even without injecting noise).

GSAT

Procedure GSAT
INPUT: a set of clauses $\alpha$, MAX-FLIPS, and MAX-TRIES
OUTPUT: a satisfying truth assignment of $\alpha$, if found
begin
  for $i$ := 1 to MAX-TRIES
    $T$ := a randomly-generated truth assignment
    for $j$ := 1 to MAX-FLIPS
      if $T$ satisfies $\alpha$ then return $T$
      $v$ := a propositional variable such that a change in its truth assignment gives the largest increase in the number of clauses of $\alpha$ that are satisfied by $T$.
      $T$ := $T$ with the truth assignment of $v$ reversed
    end for
  end for
  return "no satisfying assignment found"
end

The Search Behavior of GSAT

- In contrast to normal local search methods, we must also allow sideways movements!
- Most time is spent searching on plateaus.

State of the Art

- SAT competitions since beginning of the ´90
- Current SAT competitions (http://www.satlive.org/):
  In 2003:
  - Largest solved instances: 100,000 variables / 1,000,000 clauses
  - Smallest unsolved instances: 200 variables / 1,000 clauses
- Complete solvers are as good as randomized ones!
Concluding Remarks

- DP-based SAT solver prevail:
  - Very efficient implementation techniques
  - Good branching heuristics
  - Clause learning
- Incomplete randomized SAT-solvers
  - are good (in particular on random instances)
  - but there is no dramatic increase in size of what they can solve
  - parameters are difficult to adjust