Bipath Consistency Revisited

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Abstract. In the field of qualitative spatial and temporal reasoning combinations of constraint calculi have attracted considerable research interest in recent years. Beside combinations of spatial and temporal calculi, it is an important research topic to develop generic methods for combining calculi dealing with different spatial aspects. The prototypical example is the combination of the region connection calculus RCC8 and the point algebra first discussed by Gerevini and Renz, which allows to represent, and reason about, topological and size information about spatially extended objects. To solve constraints in this calculus, Gerevini and Renz also proposed an algorithm, the bipath consistency algorithm, which allows for deciding consistency of a given constraint network for specific sets of relations combining topology and size. In this article we will compare the "bipath consistency"-based combination method to the standard method, which is to combine calculi by generating a new calculus and applying the standard path consistency method. Gerevini and Renz's calculus combining topological and size information will serve as the running example of such combinations and also as a test case for an empirical analysis.

1 INTRODUCTION

Qualitative constraint calculi are representation formalisms which allow for efficient reasoning about continuous aspects of the world. Many of these calculi discussed in the domain of qualitative spatial reasoning can be represented as combinations of other, simpler and more compact formalisms. For example, the cardinal direction calculus [6, 10] can be seen as a specific product of the point algebra [19] with itself and the rectangle algebra [2, 3] as a twofold product of the interval algebra [1]. From a formal point of view such *orthogonal* combinations are easily definable and also partially quite well understood. Contrary to orthogonal product constructions, combinations of calculi are more interesting if the relations considered in the calculi to be integrated show semantic interdependencies. A typical example of this sort is the combination of RCC8 [16] with the point algebra, which allows for representing, and reasoning about, topological relations between regions as well as their relative sizes [8, 15]. While orthogonal combinations appear to be of interest if one can *decompose* the relations between entities of a given domain into different, logically independent aspects that can be handled in simpler formalisms, semantically interfering combinations aim at increasing the *expressiveness* of the formal language used to describe relations between objects in the domain at hand.

From a more practical point of view, semantically interfering combinations play an important role, whenever qualitative spatial information originating from different sources needs to be integrated and processed by exploiting exactly these semantic interdependencies. For example, knowing the relative sizes between objects will already restrict the topological relations that are possible between these objects, and vice versa.

With regard to non-orthogonal combinations, there exist two different possible combination strategies. Gerevini and Renz [8] proposed a method for integrating such formalisms in a *loose* way by considering constraint networks in which edges can be labeled by pairs of relations, each of which stemming from one of the combined calculi (i.e. each edge is labeled by both a topological relation and a relative size relation). For reasoning with such constraint networks (following referred to as *biconstraint networks*), Gerevini and Renz also presented an adaption of the usual path consistency algorithm (see, e.g. [13]), called *bipath consistency algorithm*, and showed that this algorithm decides satisfiability for a rather large class of biconstraint networks with topological and comparative size relations.

The second strategy is to build a new constraint language, which in general leads to a more *tight* integration. While in the bipath consistency method semantic interdependencies are repeatedly propagated in the reasoning processes in one of the component calculi, these interdependencies are exploited to define a new set of base relations and a new composition table, which is often more refined than the composition table that one obtains by taking the Cartesian product of the composition table entries of the component calculi. If such a compositional refinement is not performed reasoning in the new combined calculus (via the usual path consistency method) will just provide an upper approximation, of course.

In this paper we will investigate and compare both methods in more detail. For this Gerevini and Renz's combination of RCC8 and the point algebra will serve as our running example. We will first show that tight integrations are more expressive than loose integrations via biconstraint formalisms. Then we will report on a series of empirical tests in which we compared both methods on random instances. These results suggest that the bipath consistency approach performs well, if tractable subclasses of the component calculi are known and applied in the reasoning process. If reasoning is performed in comparatively small combined calculi or if no tractable subclasses are known for the component calculi, the method of building a new calculus may be more advantageous.

2 QUALITATIVE CONSTRAINED-BASED REASONING

Reasoning tasks in qualitative calculi are usually cast as constraint satisfaction problems over infinite, continuous domains. In order to check, for example, the consistency of a spatial description, one represents the given information as a constraint network based on an appropriate spatial calculus. Such constraint networks can be represented as a directed labelled graphs containing the variables as nodes and information about relations between two variables as labeled

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edges.

Since it is not possible to check all possible assignments to variables of the network in the given domain until one finds a solution of the network (because of the infinite domains), other techniques based on the algebraic and semantic properties of the considered calculus must be applied for testing satisfiability. The path consistency algorithm [12, 5] operates on the associated constraint graph by successively refining the labels $R_{x,y}$ (on the edge from node x to node y) via the operation $R_{x,y} \leftarrow R_{x,y} \cap (R_{x,z} \circ R_{z,y})$, where z is any third variable occurring in the network.

Since, in general, the path consistency method is not sufficient to decide consistency of constraint networks, chronological backtracking can be used, hence trying out different instantiations of the constraints containing disjunctions of base relations [9, 18]. Moreover, by using known tractable subclasses of a calculus (i.e., sets of relations closed under intersection and composition, for which the path consistency method decides consistency), one can speed up the reasoning time: instead of splitting a constraint during backtracking into base relations, one can split it into relations belonging to a tractable subclass, which reduces the branching factor of the search tree considerably [14].

3 EXAMPLE: RCC8 WITH RELATIVE SIZE CONSTRAINTS

Among other combinations, such as topology and metric size constraints, Gerevini and Renz [8] studied the combination of RCC8 with qualitative size relations (i.e. the point algebra, following referred to as $\mathcal{QS}$). The domain of this combination is the set of all *measurable* spatial regions.[2] A combined problem instance for RCC8 and $\mathcal{QS}$ is given by two sets of constraints on the same variables, such that one set of constraints uses relations only from RCC8 and the other one only relations from $\mathcal{QS}$ (*biconstraint networks*). Hence, each problem instance consists of two qualitative constraint networks on the same set of variables and each network only has constraints from one calculus. A *solution* of a biconstriant network is an assignment that simultaneously satisfies the constraints in both constraint networks. The two constraint networks are not independent, since the constraints from RCC8 and $\mathcal{QS}$ do not necessarily share a common solution. For example, Figure 1 depicts two such constraint networks that are individually but not jointly satisfiable.

As can be seen in the (not minimal) example depicted in Figure 1, not even atomic constraint networks need to share a common solution. In the example, a needs to be a non-tangential proper part of b, but a must also be larger than b, which is semantically impossible. This means that base relations from RCC8 impose constraints on the possible relations in $\mathcal{QS}$, and vice versa; e.g., if a equals b, they certainly have the same size. These constraints can be represented by so-called *interdependency tables*. An interdependency table assigns to each base relation of one calculus the *strongest entailed* relation from the other calculus. The notation $I_{\Gamma \to \Gamma'}$ will be used to refer to such interdependency maps. Figure 2 presents the interdependency tables for RCC8 and $\mathcal{QS}$. So, for instance, $I_{RCC8 \to \mathcal{QS}}(\{\text{TPP}\}) = \{<\}$. Since a relation is a disjunction of base relation, a non-base relation R entails the union of all relations entailed by the base relations from R:

$$I_{\Gamma \to \Gamma'}(R) = \bigcup_{B \in R} I_{\Gamma \to \Gamma'}(B).$$

This means that we can extend interdependency maps from base relations to arbitrary relations considered in the calculi.

Furthermore, the interdependency approach can be applied to any semantically interfering combination of calculi expressible via a biconstraint formalism.

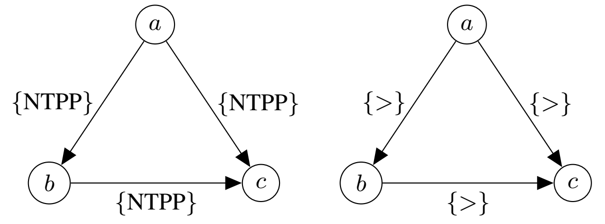

Figure 1. Two qualitative constraint networks over a domain of measurable spatial regions, the first with relations from RCC8 and the second with relations from $\mathcal{QS}$. Although both are satisfiable, they do not have a common solution.

$B \in \mathcal{QS}$		$S \in$ RCC8
$=$	$\models$	DC, EC, PO, EQ
$>$	$\models$	DC, EC, PO, TPPI, NTPPI
$<$	$\models$	DC, EC, PO, TPP, NTPP

$B \in$ RCC8		$T \in \mathcal{QS}$	$B \in$ RCC8		$T \in \mathcal{QS}$
TPP	$\models$	$<$	DC	$\models$	$*$
NTPP	$\models$	$<$	EC	$\models$	$*$
TPPI	$\models$	$>$	PO	$\models$	$*$
NTPPI	$\models$	$>$	EQ	$\models$	$=$

Figure 2. Interdependency tables for RCC8 and $\mathcal{QS}$ (cf. [8]). The symbol $*$ denotes the universal relation in $\mathcal{QS}$. Like composition table entries, lists of base relations are read disjunctively.

4 BIPATH CONSISTENCY

To decide consistency of biconstraint networks (for the combination of RCC8 and $\mathcal{QS}$) Gerevini and Renz [8] propose the bipath consistency algorithm.[3] Bipath consistency enforces algebraic closure to both constraint networks in parallel and transfers information between the two constraint network using the interdependency tables. Hence, all operations on relations are performed separately in the two calculi except for transferring information using the interdependency tables. The function `Enforce-Bipath-Consistency` given in Algorithm 1, turns two given constraint networks into *bipath consistent* networks, such that both constraint networks are algebraically closed and each constraint (x, y, R) in one constraint network is consistent with the constraint (x, y, R') in the other constraint network.

[2] Measurable sets are needed in order to adequately define *qualitative size* relations between regions in a topological space. Gerevini and Renz [8] restrict consideration to measurable sets of $\mathbb{R}^n$, e.g., spheres. For measurable entities, the three ordering relations $<, >, =$ can be used to compare entities with respect to size. These relations form a partition schema on each set of measurable regions. The resulting calculus $\mathcal{QS}$ has the same algebraic structure as the point algebra.

[3] In the light of [11] one should keep in mind that the path consistency method does not necessarily enforce path-consistent networks, but just algebraically closed ones. If one prefers the name "algebraic closure algorithm" one should use the term "bialgebraic closure algorithm" as well. In this work, however, we will use the term "bipath consistency algorithm" as established in the literature.

The algorithm is based on the path consistency algorithm by Villain and Kautz [19, 4]. Its time complexity and space complexity is still $O(n^3)$ and $O(n^2)$, respectively.

Input: Two normalized qualitative constraint networks C, C' with constraints (v_i, v_j, S_{ij}) in C, (v_i, v_j, T_{ij}) in C', $1 \le i, j \le n$ and variables $v_1, \ldots, v_n$. $\circ, \circ'$ refer to the symbolic composition operations from the two calculi Γ, Γ' and $*, *'$ to their universal relations.
Output: The algorithm returns true if bipath consistency has been enforced and false if one of the constraint networks includes the empty relation.

`Enforce-Bipath-Consistency(`C, C'`)`
  $Q \leftarrow \{ (i,j) \mid i < j \}$
  while $Q \neq \emptyset$ **do**
    select and remove a tuple (i,j) from Q
    for $k \in \{1, \ldots, n\}$: $k \neq i, k \neq j$ **do**
      if `Birevise(`i, j, k`)` = true **then**
        if $S_{ik} = \emptyset$ *or* $T_{ik} = \emptyset$ **then return** false
        else $Q \leftarrow Q \cup \{ (i,k) \}$
      if `Birevise(`k, i, j`)` = true **then**
        if $S_{kj} = \emptyset$ *or* $T_{kj} = \emptyset$ **then return** false
        else $Q \leftarrow Q \cup \{ (k,j) \}$
  return true

`Birevise(`i, k, j`)`
  if $I_{\Gamma \to \Gamma'}(S_{ik}) \cap T_{ik} = *'$ *and* $I_{\Gamma' \to \Gamma}(T_{ik}) \cap S_{ik} = *$ **then**
    return false
  if $I_{\Gamma \to \Gamma'}(S_{kj}) \cap T_{kj} = *'$ *and* $I_{\Gamma' \to \Gamma}(T_{kj}) \cap S_{kj} = *$ **then**
    return false
  old$T_{ij} \leftarrow T_{ij}$
  old$S_{ij} \leftarrow S_{ij}$
  $T_{ij} \leftarrow \big(T_{ij} \cap I_{\Gamma \to \Gamma'}(S_{ij})\big) \cap \Big(\big(T_{ik} \cap I_{\Gamma \to \Gamma'}(S_{ik})\big) \circ' \big(T_{kj} \cap I_{\Gamma \to \Gamma'}(S_{kj})\big)\Big)$
  $S_{ij} \leftarrow \big(S_{ij} \cap I_{\Gamma' \to \Gamma}(T_{ij})\big) \cap \Big(\big(S_{ik} \cap I_{\Gamma' \to \Gamma}(T_{ik})\big) \circ \big(S_{kj} \cap I_{\Gamma' \to \Gamma}(T_{kj})\big)\Big)$
  if $S_{ij} \neq$ oldS_{ij} **then**
    $T_{ij} \leftarrow \big(T_{ij} \cap I_{\Gamma \to \Gamma'}(S_{ij})\big)$
  if $S_{ij} =$ oldS_{ij} *and* $T_{ij} =$ oldT_{ij} **then**
    return false
  $T_{ji} \leftarrow T_{ij}^{-1}$
  $S_{ji} \leftarrow S_{ij}^{-1}$
  return true

Algorithm 1: The bipath consistency algorithm by Gerevini and Renz in a general form. See [8] for the original algorithm, with $I_{\Gamma \to \Gamma'}, I_{\Gamma' \to \Gamma}$ directly referring to interdependencies between $\mathcal{QS}$ and RCC8.

For bipath consistency and the combination of RCC8 with $\mathcal{QS}$, Renz and Gerevini proved the following theorem with regard to the maximal tractable subclasses of RCC8 ($\widehat{\mathcal{H}}_8$, $\mathcal{C}_8$, and $\mathcal{Q}_8$; see [17]):

Theorem 1 (cf. [8]) *The bipath consistency algorithm decides consistency for biconstraint networks over RCC8 and $\mathcal{QS}$ if all topological relations in the network are contained in the same tractable subclass (e.g $\widehat{\mathcal{H}}_8$, $\mathcal{C}_8$, and $\mathcal{Q}_8$).*

Due to the time and space complexity of the bipath consistency algorithm, it follows that biconstraint networks over RCC8 and $\mathcal{QS}$ can be decided with a time complexity of $O(n^3)$ and a space complexity of $O(n^2)$, if they contain only topological relations from one of the three maximal tractable subsets.

The bipath consistency approach has the advantage that the symbolic composition is calculated only within one calculi at a time with added information from the interdependency tables. This does not increase the complexity of the reasoning compared to the algebraic closure method. However, formal languages based on biconstraint networks are limited in terms of expressiveness. With biconstraint networks one can only express conjunctions of disjunctions of atomic expressions in one calculus; i.e., it is only possible to specify indefinite knowledge within one calculus at a time. By a biconstraint network in two variables, one could, for example, express:

$$(a < b \vee a = b) \wedge (a \text{ NTPP } b \vee a \text{ DC } b).$$

But, it is in general impossible to specify mixed indefinite knowledge such as expressed in

$$(a < b \wedge a \text{ DC } b) \vee (a = b \wedge a \text{ EC } b). \qquad (*)$$

By using biconstraints one can only express $\{<, =\}$ for the size aspect and $\{\text{DC}, \text{EC}\}$ for the topological aspect, which is equivalent to:

$$(a < b \vee a = b) \wedge (a \text{ DC } b \vee a \text{ EC } b).$$

But the latter formula would also be true in one of the following cases:

$$a < b \wedge a \text{ EC } b$$
$$a = b \wedge a \text{ DC } b$$

To express the original relation $(*)$, it is inevitable to form *tight constraint networks*, by considering the calculus that results from taking as base relation those pairs of base relations from the component calculi that are permitted by the interdependency tables. Relations in this new calculus are then just unions of such base relation pairs. In the given example, one can express $(*)$ by the set: $\{(\text{DC}, <), (\text{EC}, >)\}$.

To put this more formally, we define:

$$\mathcal{B} \;:=\; \{ (B_1, B_2) \mid B_1 \in \text{RCC8}, B_2 \in \mathcal{QS}, B_1 \in I_{\mathcal{QS} \to RCC8}(B_2), B_2 \in I_{RCC8 \to \mathcal{QS}}(B_1) \}$$

Furthermore, the `Birevise` function already induces a composition function for these new base relations in $\mathcal{B}$. Since base relations correspond to biconstraints, one can define a composition by:

$$(B_1, B_2) \circ (B_1', B_2') := \big((B_1 \circ_1 B_1') \times (B_2 \circ_2 B_2')\big) \cap \mathcal{B},$$

where $\circ_1$ and $\circ_2$ are the compositions in the original calculi RCC8 and $\mathcal{QS}$, respectively. The intersection with $\mathcal{B}$ merely removes pairs of relations that are empty. Moreover, the above defined composition is extended to arbitrary relations in the usual way:

$$\{B_1, \ldots, B_k\} \circ \{B_1', \ldots, B_l'\} = \bigcup_{i=1}^{k} \bigcup_{j=1}^{l} B_i \circ B_j'$$

Although the composition of base relations follows from the `Birevise` function, its extension to general relations differs from the results of the `Birevise` method. For example, consider the following composition for biconstraints:

$$(\{\text{TPP}, \text{TPPI}\}, \{<, >\}) \circ (\{\text{PO}\}, \{<\}) \qquad (1)$$

Both biconstraints are already refined (w.r.t. semantic interdependencies). Applying the `Birevise` function will not exclude the biconstraint $(\{\mathrm{DC}\}, \{>\})$: the relation DC is only obtained from the composition of $\{\mathrm{TPP}\}$ and $\{\mathrm{PO}\}$, not from the composition of $\{\mathrm{TPPI}\}$ and $\{\mathrm{PO}\}$. Now, if the biconstraints in (1) are written as relations using combined base relations, one obtains:

$$\{(\mathrm{TPP}, <), (\mathrm{TPPI}, >)\} \circ \{(\mathrm{PO}, <)\} \tag{2}$$

The left relation in (2) does not include $(\mathrm{TPP}, >)$, since this is not permitted by the interdependency tables. From the composition tables of RCC8 and $\mathcal{QS}$ it then follows that the base relation pair $(\mathrm{DC}, >)$ is not part of the composition result.

Since it is easy to see that all resulting base relations with the defined composition also appear in the result of the `Birevise` function on biconstraints, it immediately follows with Theorem 1:

Theorem 2 $(\widehat{\mathcal{H}}_8 \times \mathcal{P}^{\{<,=,>\}}) \cap \mathcal{B}$ *is a tractable subclass*[4] *of the qualitative constraint calculus obtained by combining base relations to pairs with entries from RCC8 and $\mathcal{QS}$. Consistency is decided by the path consistency algorithm.*

With this tight integration, we obtain a more expressive formalism which opens up more possibilities in terms of representation and reasoning, but at the same time requires more space to represent and thus more time to process, as there are more different relations. The loose integration features $(2^8 - 1) + (2^3 - 1) = 1785$ pairs of non-empty relations. Of those, 1749 represent non-empty pairs if the interdependencies are taken into account, and further only 549 distinct pairs of relations are obtained if the pairs are refined using the interdependencies. This means only 550 semantically different relations (including the empty relation) exist between objects within the loose integration. For the tight integration we obtain 14 base relations and hence $2^{14} = 16384$ relations. This begs the question how this large number of relations actually compares to the loose integration in terms of reasoning performance. In the following we report on some of our empirical results.

5 EMPIRICAL RESULTS

We compared the performance of reasoning on RCC8 and $\mathcal{QS}$ information using the two distinct approaches of tight and loose integration. To achieve this, the consistency of randomly generated constraint networks had to be decided by each approach. For this a backtracking search over constraint decompositions was performed with (a) the path consistency algorithm and (b) the bipath consistency method as forward checking. Path consistency is referred to as algebraic closure in the following.

The bipath consistency method was implemented in the Generic Qualitative Reasoner (GQR) [7], which is under active development at the Universities of Freiburg and Hildesheim. Both procedures were evaluated on randomly generated instances from the transition phase. To provide a fair comparison, the generated constraint networks used constraints that have a Cartesian product form, such that the two procedures had equivalent inputs. We used the network degree as the control parameter (cf. [15]). Hence, with n being the number of nodes and d the average degree of the graph, $\frac{n \cdot d}{2}$ edges were set to non-empty uniformly distributed relations of Cartesian product form for each instance. We found that a degree of only 2.25 already results in instances for which the probability of being consistent is around 50% for graphs with 100 nodes. A degree of 2 was used to setup all of the instances used in the following regardless of their actual network size. The benchmarks were run on a Intel Xeon processor with 3GHz and 3 gigabyte of memory.

If no tractable subclasses are used, then reasoning with the backtracking algorithm in conjunction with algebraic closure turns out to be faster than the backtracking algorithm using bipath consistency. A corresponding plot showing the average runtime of both approaches on 250 instances for each node size can be seen in Figure 3.

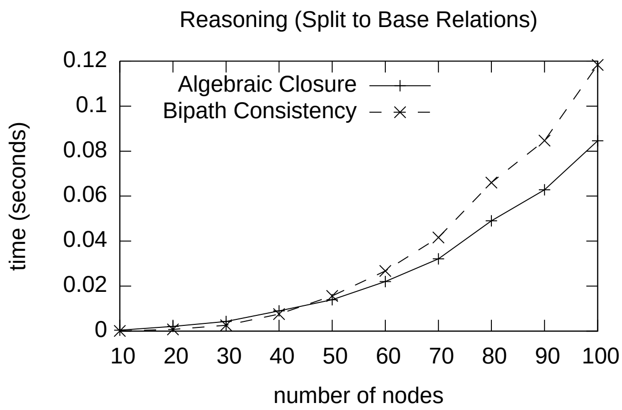

Figure 3. CPU runtime for reasoning with backtracking using algebraic closure and backtracking using bipath consistency. Relations were split to base relations.

Interestingly, the worse performance of bipath-consistency-based reasoning is entirely due to the size of the explored search space and not due to the runtime of the bipath consistency algorithm itself. This can be seen from the plot in Figure 4 showing only the runtime of algebraic closure and bipath consistency. From this, it can be concluded that the worse performance of bipath consistency is due to the weaker refinements during forward-checking performed by bipath consistency.

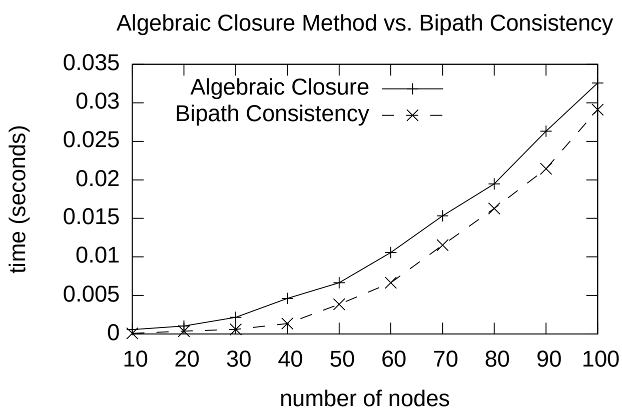

Figure 4. CPU runtime of algebraic closure and bipath consistency.

However, if the tractable subclass $(\widehat{\mathcal{H}}_8 \times \mathcal{P}^{\{<,=,>\}}) \cap \mathcal{B}$ is ex-

[4] We use the term "subclass" but do not want to imply that it is algebraically closed.

ploited, reasoning with bipath consistency outperforms all others – even if the tractable subclass is used for reasoning with the algebraic closure method as well. In this case, bipath-consistency-based reasoning benefits from two factors. Firstly, because of the compact representation it is more likely that pairs of relations are in the $(\widehat{\mathcal{H}}_8 \times \mathcal{P}^{\{<,=,>\}}) \cap \mathcal{B}$ class, whereas the algebraic closure method might refine relations further, such that they are not included in the tractable subclass anymore. Secondly, if definite information about qualitative size between two entities is given, then the topological relation already refines to a relation from $\widehat{\mathcal{H}}_8$ (cf. [8]). The corresponding plot can be seen in Figure 5. Also, reasoning with algebraic closure performs badly when used in conjunction with the tractable subclass. It is nearly two times slower than algebraic closure with splittings to base relations. A further analysis of the number of visited nodes (not given within in this article) reveals that this is due to the lack of guidance for the search. It should be noted that this is most likely due to the fact that the implementation only used an approximative decomposition of relations into relations from the tractable subclass and that no heuristic search was used in all of these benchmark runs.

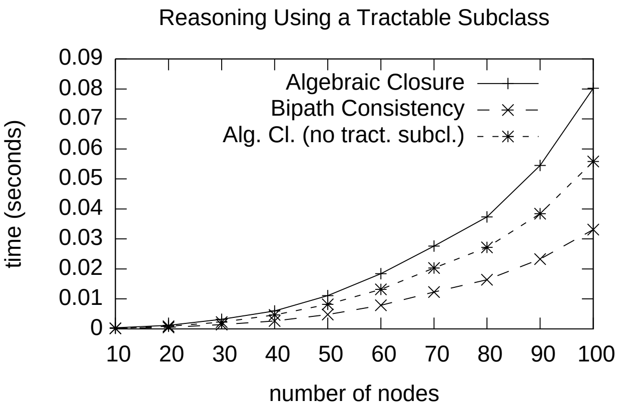

Figure 5. CPU runtime of reasoning procedures with the tractable subclass and reasoning using algebraic closure without the tractable subclass.

6 CONCLUSION

We have shown how reasoning based on the bipath consistency algorithm can be compared with reasoning using search and algebraic closure. From our empirical evaluation, it can be concluded that bipath consistency performs well, if appropriate tractable subclasses are used or if the calculus size is so large that a tight, but non-compact representation slows down other procedures. When reasoning is performed within comparatively small combined calculi or when tractable subclasses are unknown, reasoning with algebraic closure can be faster. On the other hand, application of the bipath consistency method will limit the expressiveness and performs a worse forward checking in general.

The bad performance of algebraic closure using the tractable subclass also leads to the conclusion that heuristic search methods should be used to improve reasoning performance in this case. At least for the combination of RCC8 with $\mathcal{QS}$ it seems possible that this could perform better than bipath consistency with tractable subclasses.

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